Continuum mechanics of two-phase porous media

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Preface

The present text was developed during the course of two Ph.D. courses on porous materials modeling given at the Department of Applied Mechanics, Chalmers. The main purpose of this course is to give an up-to-date account of the fundamental continuum mechanical principles pertinent to the theory of porous materials considered as mixture of two constituents. The idea of the course is to provide a framework for the modeling of a “solid” porous material with compressible and incompressible fluid phases. As to constitutive modeling, we restrict to hyper-elasticity and the ordinary Darcy model describing the interaction between the constituents. Computational procedures associated with the nonlinear response of the coupled two-phase material will be emphasized. The present text is focusing on the general description of kinematics and material models for FE-modeling of large deformation problems.

The first course was given for the first time during the autumn 2006. The text has continuously been revised during the period 2006–September 2012.

Göteborg in September 2012

Ragnar Larsson
Chapter 1

Introduction

1.1 Background

The area of multiphase materials modeling is a well-established and growing field in the mechanical scientific community. Porous materials are encountered in a broad spectrum of engineering applications. The mechanics of porous media is nowadays relevant in disciplines as varied as soil mechanics, geophysics, biomechanics, material science, and in the simulation of industrial processes, to mention only a few. Along with the increasing capacity of computers, attention is also continuously directed towards new applications.

There has been a tremendous development in recent years including the conceptual theoretical core of multiphase materials modeling, the development of computational methodologies as well as experimental procedures. Applications of the theory concern biomechanics, modeling of structural foams, process modeling of composites, soils of geomechanics, road mechanics etc. Specific related issues concern modeling of: solid-fluid interaction, compressible-incompressible fluids/solids including phenomena like consolidation, compaction, erosion, growth, wetting, drying etc.

A characteristic feature of the theories of porous media is the adoption of a macroscopic scale. This means that one does not consider the micro-problem of fluid(s) running through a complex pore structure, but rather accounts for this phenomenon in an "average sense". Although the modeling of porous materials has interested investigators for a long time, the theory and especially its rational treatise in particular applications, like e.g. composites process modeling (involving macro and micro wet out problems as well as free surface problems), is a subject for intense research, e.g. refs. [35], [34].

There is a quite extensive background to the theoretical development and numerical treatment of consolidation/porous media theory. Without the intention to be complete, it is noted that a contribution to this subject is due to de Boer [13], where also early (18th
and 19th centuries) developments are discussed. In particular, the very first steps in the development of modern soil mechanics were taken by Karl von Terzaghi in Vienna in the early years of this century. He was the first to tackle the problem of a deformable fluid-filled solid and stated his famous 1D consolidation equation in 1923. It is a differential equation in the excess pore pressure and is still used for approximate settlement analyses. Terzaghi also made progress in investigating the strength of soil when he 1936 formulated explicitly the now widely used and well-known effective stress principle, cf. section 3.4.5. This principle is of prime importance in Geomechanics.

Biot [7], (1941) was one of the first who presented a formulation for consolidation in 3D. This theory was later refined and developed continuously by himself in a series of publications e.g. Biot [8], [9], [10], [11]. In this theory, the basic static variables are the total stress and the fluid pressure. The associated kinematic variables are the strain in the skeleton and the variation of fluid content (volume change of fluid per unit volume of mixture). There exists several extensions and generalizations of his theory in the literature. A quite modern formulation, which may be categorized as an extended 'Biot framework', is given in the comprehensive book by Coussy [22] (cf. also Coussy [21]). For early advanced representative numerical formulations see Small et al. [40], the Ph.D. thesis by Runesson [38], Carter et al. [20], Zienkiewicz and Shiomi [45] and Armero [5].

While the 'Biot theory' was developed primarily within soil mechanics, aimed at the mixture of solid skeleton and pore fluid(s), it has been also a development of mixture theory in a broader sense. The basis of this theory is the abstraction that the constituents of a mixture can be modeled as superimposed continua. The usual continuum mechanics principles are then used for the individual constituents, with the proper interaction terms included. It is then useful to adopt the important concept that the equations of balance of the individual constituents, when summed over all the constituents, should yield the equations of balance for the mixture as a whole. This concept provides constraints to the introduced interaction terms, and it was adopted as a principle in Truesdell [42]. Examples of early contributions to the classical theory are e.g. Truesdell and Toupin [43], Kelly [30], Green and Naghdi [29], Eringen and Ingram [27]. Later, the need to describe the behavior of the actual material constituting the mixture (and also to describe immiscible mixtures) gave rise to the introduction of the volume fraction concept, e.g. Goodman and Cowin [28], Bowen [18], [19] and Ehlers [24].

A comprehensive survey of mixture theories up to its date can be found in Bedford and Drumheller [6]. See also the historical review article by de Boer [3], and de Boer and Ehlers [14] concerning the concept of volume fractions. Applied to soil mechanics, some early numerical formulations are found in Sandhu and Wilson [39]. Recent contributions are due to e.g. Borja et al. [16], [17], Diebels and Ehlers [23]. Although Biot obviously developed his theory independent of mixture theory, it may in a sense be apprehended as a specific type of mixture theory; specialized to the situation of fluid-filled porous solid.
To this end, the thermodynamic Lagrangian formulation arrived at by Biot [11] marks one difference between the Biot theory and the mixture theory in general. The Lagrangian thermodynamic formulation by Biot was reworked in Coussy [21], [22] via the introduction of the "open thermodynamic continuum". This means to choose the solid as a reference constituent, which consequently embodies the reference cell. The notion "open" refers to the fact that the solid content is conserved in this cell whereas fluid may be exchanged with the exterior. Concerning this discussion it is argued, cf. Biot [10], [11], that the Eulerian nature of mixture theories makes them well suited for fluid mixtures but not for situations where a typically history dependent and/or structured porous solid is present.

On the basis of mixture theory combined with the volume fraction concept, we propose in this text a formulation for large elastic deformations in combination with a compressible solid and fluid phases. The thermodynamic formulation is inspired by the "open thermodynamic continuum" as introduced by Coussy [21]. Put differently, this results in a Lagrangian description of the skeleton and a modified Eulerian description of the fluid(s). It is a modified Eulerian description since the position of the reference cell is not fixed in space but follows the skeleton movement. By focusing on essential nonlinearities like liquid permeability (as induced large deformation) and the free surface problem, the developed biphasic continuum mechanical models accounts for the relevant physical properties. For the geometrically linear case, a quite advanced model was proposed by Ehlers et al. [26] for modeling deformation in partially saturated soils. The model by [26] is based on a triphasic formulation where the unsaturated soil is considered as a materially incompressible solid skeleton saturated by two viscous pore-fluids. Of particular relevance for these developments is the modeling of fluid compressibility, and, related to that, the production of liquid in the unsaturated void space. Issues relating to the modeling of compressible constituents have previously been considered by Bowen [19], Svendsen and Hutter [41], de Boer [13]. In particular, with respect to the compressible fluid phase formulation, where the response of the fluid pressure is modeled in terms of the amount of dispersed air in the liquid, we refer to Ehlers and Blome [25]. Various approaches have been used, such as multiplicative decomposition of the phase deformation in real and remaining parts; cf. de Boer [13]. In Svendsen and Hutter [41] emphasis is placed on a general thermodynamic formulation for a multiphase mixture, which considers the entropy inequality supplemented by the proper constraints of saturation and mass balance, cf. Liu [37]. Another approach is to formulate mass balance in terms of logarithmic compaction strain measures, which may be used to specify the compressibility of the different phases via the entropy inequality, cf. Larsson and Larsson [33], [36].

We note that the difference between Biot- and mixture theory formulations appears not to be completely agreed upon in the literature. For instance, Zienkiewicz et al. [44] writes "... Later it became fashionable to derive the equations in the forms of so called mixture theories ... If correctly used, the mixture theory establishes of course identical equations (to Biot-like equations, the author)...". This viewpoint is to be contrasted with
de Boer [13], who makes a clear distinction between these approaches and writes "... The mixture theory restricted by the concept of volume fractions (porous media theory) yields the most consistently developed frame to treat liquid-saturated porous solids. Other approaches to investigate such bodies are very often based on partly obscure assumptions ...".

1.2 Organization of lectures

The course material is defined by the content of this text plus additional literature references given during the course. An outline of the course is given by:

- Introduction and applications of the porous media theory.
- The concept of a two-phase mixture: Volume fractions, Effective mass, Effective velocities, Homogenized stress.
- Kinematics of a two-phase continuum.
- Conservation of mass. One-phase material, Two-phase material, Mass balance of fluid phase in terms of relative velocity, Mass balance in terms of internal mass supply, Mass balance - final result.
- Conservation of momentum changes and energy
  - Momentum: Total format, Individual phases and transfer of momentum change between phases.
- Conservation of energy (cont’d) and Entropy inequality.
  - General approach (effective free energy), Localization, Effective drag (or interaction) force.
- Constitutive relations:
  - Effective stress response, Solid-fluid interaction, Solid densification - reduction of a three phase model, Gas densification - the ideal gas law
- Summary – Balance relations for different types of porous media.
  - Classical incompressible solid-fluid medium, Compressible solid-fluid medium, Compressible solid-gas medium. Restriction to small deformations – Compressible solid-fluid medium
1.3. COURSE WORK

- Modeling of effective solid phase (Hyper-elasticity), Darcy interaction, Issue of incompressibility, boundary value problem.

  • Computational aspects: discretization, set of non-linear FE equations, solution of coupled problem (monolithic/staggered solution techniques). Assignment: specific model, cont’d.

  • Summary of the course: dugga.

1.3 Course work

The key to the course is to work with the “theory questions” (initiating each section) involving derivation of continuum mechanical relations related to the modeling of porous materials and “computer implementation” of a chosen specific model. In addition an assignment problem is to be defined and completed for the total credit of the course.
CHAPTER 1. INTRODUCTION
Chapter 2

The concept of a two-phase mixture

In this chapter we make a preliminary discussion of a solid-fluid mixture considered as an homogenized two-phase material. It is then common to consider the saturated porous medium as superposed, in time and space, of the constituting phases. This means that, although neither the solid skeleton nor the fluid is continuous in space, they are both considered as continua. Consequently, it is necessary to adopt a macroscopic view for the description of local quantities such as the stress, the deformation gradient etc. To this end we introduce the Representative Volume Element (RVE) with volume $V$, as in Fig. 2.1 with the involved solid-fluid constituents on a sub-scale and the corresponding homogenization on the macro-scale. This representative volume must include material enough for it to be representative of the studied macroscopic behavior, but at the same time, it must be small enough to represent the local dependence of the averaged quantities. Clearly, this view confirms the existence of a "scale" of the elementary volume, which should be sufficiently small as compared to the scale of the intended application. Under these circumstances, one considers the field quantities to be continuous point-wise; for instance, the deformation of the skeleton is defined in every material point and is continuous between two neighboring points. These ideas are summarized in Fig. 2.1.

In order to arrive at the interpretation of the involved quantities in the macroscopic continuum level considered as volume averages, a brief ad-hoc description is given below of the homogenization of the solid-fluid micro-constituents. We emphasize that in the final continuum formulation of the two-phase mixture, we refrain from the detailed consideration of the relationships between the constituents.

The following theory questions define the line of developments of the following sections:
CHAPTER 2. THE CONCEPT OF A TWO-PHASE MIXTURE

1. Define and discuss the concept of volume fractions in relation to the micro-constituents of a two-phase mixture of solid and fluid phases related to an RVE of the body.

2. Define the effective mass in terms of intrinsic and bulk densities of the phases from equivalence of mass. Discuss the issue of (in)compressibility of the basis of this discussion.

3. Define the effective (representative) velocities of the solid and fluid phases related their micromechanical variations across an RVE. Discuss also the issue of “direct averaging”.

4. Based on the quite general result of stress homogenization of a one-phase material, generalize the result to the two-phase situation. Discuss the partial stresses and their relation to intrinsic stresses and micro-stress fields.

2.1 Volume fractions

To start with, let us consider the constituents homogenized with respect to their volume fractions RVE in Fig. 2.1. To this end, we introduce the macroscopic volume fractions $n^\alpha[x,t]$ as the ratio between the local constituent volume and the bulk mixture volume, i.e. $n^s = \frac{V^s}{V}$ for the solid phase and $n^f = \frac{V^f}{V}$ for the fluid phase. Of course, to ensure that each control volume of the solid is occupied with the solid/gas mixture, we have the saturation constraints

$$n^s + n^f = 1$$
$$0 \leq n^s \leq 1, \quad 0 \leq n^f \leq 1$$
2.2. EFFECTIVE MASS

We may thus formulate the volume $V$ of the RVE in terms of the volume fractions as in the sequel

\[(2.2)\quad V = V^s + V^f = \int_{B^s} dv + \int_{B^f} dv = \int_B n^s dv + \int_B n^f dv = \int_B (n^s + n^f) dv = \int_B dv\]

2.2 Effective mass

Following the discussion concerning the volume fractions, let us apply the principle of “mass equivalence” to the RVE (with volume $V$) in Fig. 2.1. By mass equivalence between the micro- and the macroscopic mass, the total mass $M$ of the RVE may be formulated as

\[(2.3)\quad M = \int_{B^s} \rho^s_{\text{mic}} dv + \int_{B^f} \rho^f_{\text{mic}} dv = \int_B n^s \rho^s_{\text{mic}} dv + \int_B n^f \rho^f_{\text{mic}} dv = \int_B (n^s \rho^s_{\text{mic}} + n^f \rho^f_{\text{mic}}) dv = \int_B (n^s \rho^s + n^f \rho^f) dv\]

where the *macroscopic intrinsic* densities (associated with each constituent are denoted $\rho^s$ and $\rho^f$) we introduced in the last equality. According to the Principle of Scale Separation (P.S.S.), i.e. that the involved *macroscopic* quantities can be considered constant across the RVE, (or in other words the subscale is “small enough” cf. Toll [2]), we now state the relationship between the microscopic and the macroscopic fields as

\[(2.4)\quad M = \int_{B^s} \rho^s_{\text{mic}} dv + \int_{B^f} \rho^f_{\text{mic}} dv = \int_B (n^s \rho^s + n^f \rho^f) dv \overset{\text{P.S.S.}}{=} \left( n^s \rho^s + n^f \rho^f \right) V\]

whereby we obtain the averages

\[(2.5)\quad n^s \rho^s = \frac{1}{V} \int_{B^s} \rho^s_{\text{mic}} dv \Rightarrow \rho^s = \frac{1}{V^s} \int_{B^s} \rho^s_{\text{mic}} dv
\]

\[n^f \rho^f = \frac{1}{V} \int_{B^f} \rho^f_{\text{mic}} dv \Rightarrow \rho^f = \frac{1}{V^f} \int_{B^f} \rho^f_{\text{mic}} dv\]

It may be noted that the intrinsic densities relate to the issue of compressibility (or incompressibility) of the phases. For example, in the case of an incompressible porous mixture, the intrinsic densities are stationary with respect to their reference configurations, i.e. $\rho^s = \rho^s_0$, $\rho^f = \rho^f_0$. Let us also introduce the bulk density per unit bulk volume $\hat{\rho} = n^\alpha \rho^\alpha$, whereby the saturated density $\hat{\rho} = \frac{M}{V}$ becomes

\[(2.6)\quad \hat{\rho} = \hat{\rho}^s + \hat{\rho}^f\]
CHAPTER 2. THE CONCEPT OF A TWO-PHASE MIXTURE

2.3 Effective velocities

To motivate the effective velocities on the macroscale, we consider the equivalence of momentum $P$ produced by micro fields and the corresponding effective fields on the macroscale of the RVE in Fig. 2.1. This is formulated as

$$P = \int_{B_s} \rho_s^{\text{mic}} v_s^{\text{mic}} dv + \int_{B_f} \rho_f^{\text{mic}} v_f^{\text{mic}} dv = \int_{B} \left( n^s \rho_s^{\text{mic}} v_s^{\text{mic}} + n^f \rho_f^{\text{mic}} v_f^{\text{mic}} \right) dv \overset{P.S.S.}{=} (n^s \rho^s v^s + n^f \rho^f v^f) V$$

where (again) the last equality follows from the principle of scale separation. From this relation we may choose to specify the effective properties as:

$$\int_{B_s} \rho_s^{\text{mic}} v_s^{\text{mic}} dv - \rho^s v^s (n^s V) = 0 \quad \int_{B_f} \rho_f^{\text{mic}} v_f^{\text{mic}} dv - \rho^f v^f (n^f V) = 0$$

corresponding to the averages

$$\rho^s v^s = \frac{1}{V_s} \int_{B_s} \rho_s^{\text{mic}} v_s^{\text{mic}} dv$$

$$\rho^f v^f = \frac{1}{V_f} \int_{B_f} \rho_f^{\text{mic}} v_f^{\text{mic}} dv$$

Hence, the effective velocity fields $v^s$ and $v^f$ are considered as mean properties of the momentum of the respective constituents scaled with the intrinsic densities $\rho^\alpha$. If we in addition assume that $\rho_s^{\text{mic}}, \rho_f^{\text{mic}}$ are completely independent (or uncorrelated) we obtain the direct averaging:

$$v^s = \frac{1}{V_s} \frac{1}{\rho^s} \int_{B_s} \rho_s^{\text{mic}} v_s^{\text{mic}} dv = \frac{1}{V_s} \int_{B_s} v_s^{\text{mic}} dv, v^f = \frac{1}{V_f} \int_{B_f} v_f^{\text{mic}} dv$$

where (due to the assumed uncorrelation) it was used that

$$\int_{B_s} \rho_s^{\text{mic}} v_s^{\text{mic}} dv = V_s \int_{B_s} \rho_s^{\text{mic}} \frac{dv}{V_s} = V_s \int_{B_s} \rho_s^{\text{mic}} \frac{dv}{V_s} = \frac{1}{V_s} \int_{B_s} \rho_s^{\text{mic}} dv \int_{B_s} v_s^{\text{mic}} dv$$

2.4 Homogenized stress

Given the existence of total surface forces in a cut of porous media, the definition of the total (Cauchy) stress does not differ from that of the stress in a standard monophasic
2.4. HOMOGENIZED STRESS

continuum. From homogenization theory and micromechanics of solid materials, cf. Toll [2], let us consider the total stress as the volumetric mean value over a representative volume element with the volume $V$ as

$$\overline{\sigma} = \frac{1}{V} \int_B \sigma_{\text{mic}} dV$$  \hspace{1cm} (2.13)$$

where $\sigma_{\text{mic}}$ is the micromechanical variation of the stress field (comprising both the solid and fluid phases) within the RVE in Fig. 2.1. However, this total stress does not separately take into account the microscopic stresses, which are related to the solid matrix and to the pore fluid. It rather represents a weighted average of the micro stresses with respect to the elementary volume, cf. eq. 2.13.

In order to link the macroscopic total stress tensor to the individual constituent stresses, we are led to introduce the partial stresses. Physically, this means that the macroscopic surface forces, which are equilibrated at the macroscopic level by the total stress vector, are equilibrated at the level just below by the averaged stress vector of the material constituents. Obviously, the partial stress in a constituent reflects the part of the total stress that is carried by this particular constituent in a typical cut of the porous medium. We thus generalize this result in eq. 2.13 to the situation of a two-phase mixture of solid $s$ and fluid $f$ phases where the total (homogenized) stress is obtained as the mean value

$$\overline{\sigma} \overset{\text{def}}{=} \frac{1}{V} \left( \int_{B_s} \sigma_{s,\text{mic}} dV + \int_{B_f} \sigma_{f,\text{mic}} dV \right) = \sigma^s + \sigma^f$$  \hspace{1cm} (2.14)$$

where the introduced homogenized (macroscopic) stresses $\sigma^s$ and $\sigma^f$ are named the partial stresses of the respective solid and fluid constituents. The averaging of these stresses is then defined as

$$\sigma^s = \frac{1}{V_s} \int_{B_s} \sigma_{s,\text{mic}} dV , \quad \sigma^f = \frac{1}{V_f} \int_{B_f} \sigma_{f,\text{mic}} dV$$  \hspace{1cm} (2.15)$$

Note that $\sigma^s$ and $\sigma^f$ represent the homogenized stress response of the constituents, which can be related to the intrinsic stresses upon introducing the fractions $n^s = \frac{V_s}{V}$ of the solid phase and $n^f = \frac{V_f}{V}$ of the fluid phase, as defined in (2.11). The intrinsic stresses are then defined via

$$\sigma^s = \frac{1}{V_s} \frac{V_s}{V} \int_{B_s} \sigma_{s,\text{mic}} dV = n^s \frac{1}{V_s} \int_{B_s} \sigma_{s,\text{mic}} dV = n^s \sigma_{s,\text{in}} , \quad \sigma^f = n^f \sigma_{f,\text{in}}$$  \hspace{1cm} (2.16)$$

with

$$\sigma_{s,\text{in}} = \frac{1}{V_s} \int_{B_s} \sigma_{s,\text{mic}} dV , \quad \sigma_{f,\text{in}} = \frac{1}{V_f} \int_{B_f} \sigma_{f,\text{mic}} dV$$  \hspace{1cm} (2.17)$$
As an example, let us consider the important special case (considered later on in this course) of the assumption of an ideal fluid where the intrinsic stress response is defined by the intrinsic fluid pressure $p$ as

\begin{equation}
\sigma_{in}^{f} = -p \mathbf{1} \Rightarrow \sigma^{f} = -n^{f} p \mathbf{1}
\end{equation}

where the last expression defines the homogenized partial fluid stress in the case of an ideal fluid.
Chapter 3

A homogenized theory of porous media

In the following section we outline the relevant kinematics of pertinent to a two phase continuum. The following theory questions are addressed:

1. Define and discuss the formulation of the kinematics of a two-phase mixture. Introduce the different types of material derivatives and formulate and discuss the velocity fields of the mixture.

2. Prove the formula: \[ \dot{J} = J \nabla \cdot \mathbf{v} \]

3.1 Kinematics of two phase continuum

Let us in the following consider our porous material as a homogenized mixture between solid and fluid (which may be a liquid or a gas depending on the application) phases as motivated in the previous section. To this end, we denote the phases $s$ or $f$, where $s$ stands for the solid phase, whereas $f$ stands for the fluid phase. The representation of the porous medium as a mixture of constituents, implies that each spatial point $x$ of the current configuration at the time $t$ are simultaneously occupied by the material particles $X^\alpha$. We emphasize that the constituents $s, f$ relate to different reference configurations, i.e. $X^s \not= X^f$, cf. Fig. 3.1. During the deformation these “particles” move to the current configuration via individual deformation maps defined as

\[ x = \varphi \left[ X^s \right] = \varphi^f \left[ X^f \right] \]  

As to the associated velocity fields we have in view of the deformation maps $\varphi$ and $\varphi^f$ the relations

\[ v^s = \frac{D^s \varphi[X]}{Dt} = \dot{\varphi}[X], \quad v^f = \frac{D^f \varphi^f[X^f]}{Dt} \]
where \( D_s \cdot /Dt = \cdot \) denotes the material derivative with respect to the solid reference configuration, whereas \( D_f \cdot /Dt \) denotes the material derivative with respect to the fluid reference configuration. Please note that the “dot derivative” thus denotes the material derivative with respect to the solid reference configuration. We remark that indeed \( v^s \neq v^f \) in general. Hence, let us already at this point introduce the relative velocity between the phases \( v^r = v^f - v^s \). Related to this, we shall subsequently also consider the Darcian velocity \( v^d = n^f v^r \).

Figure 3.1: Schematic of basic continuum mechanical transformations for a mixture material.

As alluded to above, we focus our attention to the solid reference configuration, and to simplify the notation we set \( B_0 = B_0^s \). Additionally, we set \( X = X^s \) and \( v = v^s \). Hence, the mapping \( x = \varphi[X] \) characterizes the motion of the solid skeleton. In accordance with standard notation, we consider the deformation gradient \( F \) and its Jacobian \( J \) associated with \( X^s \) defined as

\[
F = \varphi \otimes \nabla \quad \text{with} \quad J = \det[F] > 0
\]

### 3.2 Conservation of mass

In this section we outline conservation of mass pertinent to a two phase continuum. We start out with the standard one-phase material and generalize the result into the two-phase case. The following theory questions are addressed:
3.2. CONSERVATION OF MASS

1. Formulate the idea of mass conservation pertinent to a one-phase mixture. Generalize this idea of mass conservation to the two-phase mixture. Discuss the main results in terms of the relative velocity between the phases.

2. Describe in words (and some formulas if necessary) the difference between the material derivative $\dot{\rho} = D\rho[\mathbf{x}, t]/Dt$ and the partial derivative $\partial \rho[\mathbf{x}, t]/\partial t$, where $\rho$ is the density of the material.

3. Formulate the mass balance in terms of internal mass supply. Discuss a typical erosion process.

4. Formulate the mass balance in terms of the compressibility strains. Set of the total form of mass balance in terms of the saturation constraint. In view of this relationship, introduce the issue of incompressibility/compressibility of the phases in terms of the compressibility strains.

3.2.1 One phase material

To warm up for the subsequent formulation of mass conservation of a two-phase material, let us consider the restricted situation of a one-phase material in which case the total mass of the solid may be written with respect to current $\mathcal{B}$ and reference $\mathcal{B}_0$ configurations as

$$\mathcal{M} = \int_{\mathcal{B}} \rho dv = \int_{\mathcal{B}_0} \rho J dV$$

where in the last equality we used the substitution $dv = JdV$.

The basic idea behind the formulation of mass conservation is that the mass of the particles is conserved during deformation, i.e.

$$m_0[\mathbf{X}] = m[\varphi[\mathbf{X}]] \iff \rho_0 dV = \rho J dV$$

where it was used that $m = \rho dv$.

Let us next apply this basic principle to our one-phase solid. First, consider the conservation of mass from the direct one-phase material written as:

$$\frac{D\mathcal{M}}{Dt} = \frac{D}{Dt} \int_{\mathcal{B}} \rho dv = \int_{\mathcal{B}_0} \left( \dot{\rho} J dV + \rho \dot{J} dV \right) = \int_{\mathcal{B}_0} \dot{\rho} J + \dot{\rho} J = 0$$

or in localized format we obtain

$$\dot{J} = \nabla \cdot \mathbf{v} = \int_{\mathcal{B}_0} \left( \dot{\rho} + \rho \mathbf{v} \cdot \nabla \right) J dV = 0$$

$$\dot{\mathcal{M}} = \left( \dot{\rho} + \rho \nabla \cdot \mathbf{v} \right) J = 0$$
3.2.2 Two phase material

Consider next the situation of two phases of volume fractions \( n^s \) for the solid phase and \( n^f \) for the fluid phase. The mass conservation of the two-phase material is now expressed with respect to the conservation of mass for the individual phases, so that the total mass \( \mathcal{M} \) is expressed as

\[
\mathcal{M} = \mathcal{M}^s + \mathcal{M}^f = \int_{\mathcal{B}_s} \rho^s dv + \int_{\mathcal{B}_f} \rho^f dv = \int_{\mathcal{B}} n^s \rho^s dv + \int_{\mathcal{B}} n^f \rho^f dv = \int_{\mathcal{B}_0} M^s dV + \int_{\mathcal{B}_0} M^f dV
\]

where \( M^s \) and \( M^f \) are the solid and fluid contents, respectively, defined as

\[
M^s = J n^f \rho^f, \quad M^f = J n^s \rho^s
\]

We thus express the mass conservation for the solid and fluid phases in turn as: The balance of mass now follows for the solid phase as

\[
\frac{D}{Dt} \int_{\mathcal{B}_s} \rho^s dv = \frac{D}{Dt} \int_{\mathcal{B}} n^s \rho^s dv = \frac{D}{Dt} \int_{\mathcal{B}_0} J \dot{\rho}^s dV = \int_{\mathcal{B}_0} \dot{M}^s dV = 0
\]

where we used the dot to indicate the rate with respect to the fixed solid reference configuration. This leads to the condition

\[
\dot{M}^s = J \dot{\rho}^s + \dot{\rho}^s J = J \left( \dot{\rho}^s + \rho^s \nabla \cdot v \right) = 0
\]

(3.11)

Hence, the stationarity of the solid content is a representative of mass conservation of the solid phase.

The balance of mass now follows for the fluid phase as

\[
\frac{D}{Dt} \int_{\mathcal{B}_f} \rho^f dv = \frac{D}{Dt} \int_{\mathcal{B}} n^f \rho^f dv = \frac{D}{Dt} \int_{\mathcal{B}_0} J \dot{\rho}^f dV = \int_{\mathcal{B}_0} \dot{M}^f dV = \int_{\mathcal{B}} \left( \frac{D}{Dt} \dot{\rho}^f + \dot{\rho}^f \nabla \cdot v^f \right) dV = 0
\]

(3.12)

where the “PF” and the “PB” denotes “Push-Forward” or “Pull-Back” operations. It should be noted that in the PB-operation we consider the fluid reference configuration \( B_0^f \) whereby we have the dependencies \( \dot{\rho}^f = \dot{\rho}^f [\varphi^f [X^f], t] = \dot{\rho}^f [X^f, t] \). Finally, pull-back to the solid reference configuration yields:

\[
\int_{B_0} \left( \frac{D}{Dt} \dot{\rho}^f + \dot{\rho}^f \nabla \cdot v^f \right) J dV = 0
\]

(3.13)
3.2. CONSERVATION OF MASS

By localization the condition for mass balance of the fluid phase is finally obtained as

$$\frac{D \hat{\rho}^f}{Dt} + \hat{\rho}^f \nabla \cdot \mathbf{v}^f = \frac{\partial \hat{\rho}^f}{\partial t} + (\nabla \hat{\rho}^f) \cdot \mathbf{v}^f + \hat{\rho}^f \nabla \cdot \mathbf{v}^f = 0$$  \hspace{1cm} (3.14)

### 3.2.3 Mass balance of fluid phase in terms of relative velocity

Upon introducing the relative velocity \( \mathbf{v}^r = \mathbf{v}^f - \mathbf{v} \) the material time derivative pertinent to the fluid phase may be rewritten as

$$\frac{D \hat{\rho}^f}{Dt} = \frac{\partial \hat{\rho}^f}{\partial t} + (\nabla \hat{\rho}^f) \cdot \mathbf{v} + (\nabla \hat{\rho}^f) \cdot \mathbf{v}^r$$  \hspace{1cm} (3.15)

whereby the balance of mass for the fluid phase in (3.14) becomes

$$\hat{\rho}^f + \hat{\rho}^f \nabla \cdot \mathbf{v}^f + (\nabla \hat{\rho}^f) \cdot \mathbf{v}^r + \hat{\rho}^f \nabla \cdot \mathbf{v} + (\nabla \hat{\rho}^f) \cdot \mathbf{v}^r = \hat{\rho}^f + \hat{\rho}^f \nabla \cdot \mathbf{v} + \nabla \cdot (\hat{\rho}^f \mathbf{v}^r) = 0$$  \hspace{1cm} (3.16)

It is also of interest to consider the fluid mass balance in terms of the fluid content defined as

$$M^f = J \hat{\rho}^f \Rightarrow \dot{M}^f = J \left( \hat{\rho}^f + \hat{\rho}^f \nabla \cdot \mathbf{v} \right)$$  \hspace{1cm} (3.17)

whereby the balance relation (3.14) is formulated as

$$\dot{M}^f + J \nabla \cdot (\hat{\rho}^f \mathbf{v}^r) = 0$$  \hspace{1cm} (3.18)

We thus conclude that the mass balance of the fluid phase may be related to the solid material via the introduction of the relative velocity. Clearly, the special situation of synchronous motion of the phases, i.e. \( \mathbf{v}^r = 0 \), corresponds to stationarity of both \( M^s \) and \( M^f \). This condition is also sometimes named the “local undrained case”.

### 3.2.4 Mass balance in terms of internal mass supply

We extend the foregoing discussion by introducing the exchange of mass between the phases via the internal mass production/exclusion terms \( G^s \) and \( G^f \), which define the rates of increase of mass of the phases due to e.g. an erosion process or chemical reactions. These are in turn related to the local counterparts \( g^s \) and \( g^f \) defined as

$$G^s = \int_B g^s \, dv = \int_{B_0} J g^s \, dV, \quad G^f = \int_B g^f \, dv = \int_{B_0} J g^f \, dV$$  \hspace{1cm} (3.19)
Hence, we may extended the balance of mass between the individual solid and fluid phases as:

\[
\frac{D_s}{Dt} M^s = G^s \iff \int_{B_0} \dot{M}^s dV = \int_{B_0} J g^s dV
\]

(3.20)

\[
\frac{D_f}{Dt} M^f = G^f \iff \int_{B_0} \left( \dot{M}^f + J \nabla \cdot (\hat{\rho}^f v^f) \right) dV = \int_{B_0} J g^f dV
\]

(3.21)

Let us in addition assume that the total mass total mass is conserved i.e. we have the condition for mass production/exclusion

\[
\frac{D_s}{Dt} M^s + \frac{D_f}{Dt} M^f = 0 \Rightarrow G^s + G^f = 0 \Rightarrow g^s + g^f = 0
\]

(3.22)

Clearly, the last expression represents the case of an erosion process where “the first phase takes mass from the second phase”. The mass balance relationship for the individual phases thus becomes

\[
\dot{M}^s = J g^s = -J g^f
\]

\[
\dot{M}^f + J \nabla \cdot (\hat{\rho}^f v^f) = J g^f
\]

(3.23)

where it may be noted that erosion normally means \(g^f = -g^s > 0\) corresponding to the situation that the fluid phase “gains” mass from solid phase.

### 3.2.5 Mass balance - final result

Let us summarize the discussion of mass balances in terms of the total mass conservation as expressed in (3.23). Hence, in view of (3.23) we may formulate the local condition for total mass balance as

\[
\dot{M}^s + \dot{M}^f + J \nabla \cdot (\hat{\rho}^f v^f) = 0
\]

(3.24)

where we note that the solid and fluid contents may be expanded in terms of the volume fractions as

\[
M^s = J \hat{\rho}^s \Rightarrow \dot{M}^s = J \left( \dot{\hat{\rho}}^s + \hat{\rho}^s \nabla \cdot v \right) = J \hat{\rho}^s \left( \dot{n}^s + n^s \nabla \cdot v + n^s \frac{\dot{\rho}^s}{\rho^s} \right)
\]

(3.25)

\[
M^f = J \hat{\rho}^f \Rightarrow \dot{M}^f = J \left( \dot{\hat{\rho}}^f + \hat{\rho}^f \nabla \cdot v \right) = J \hat{\rho}^f \left( \dot{n}^f + n^f \nabla \cdot v + n^f \frac{\dot{\rho}^f}{\rho^f} \right)
\]

(3.26)
In addition, let us next introduce the logarithmic compressibility strains $\epsilon_s^v$ for the solid phase densification and $\epsilon_f^v$ for the fluid phase densification expressed in terms of intrinsic densities $\rho^s$ and $\rho^f$ as

\[
\epsilon_s^v = -\log \left( \frac{\rho^s}{\rho_0^s} \right) \Rightarrow \dot{\epsilon}_s^v = -\frac{\dot{\rho}^s}{\rho^s} \Rightarrow \dot{\rho}^s = -\dot{\epsilon}_s^v
\]

(3.27)

\[
\epsilon_f^v = -\log \left( \frac{\rho^f}{\rho_0^f} \right) \Rightarrow \dot{\epsilon}_f^v = -\frac{\dot{\rho}^f}{\rho^f} \Rightarrow \dot{\rho}^f = -\dot{\epsilon}_f^v
\]

(3.28)

The mass balance relations now becomes in the compressibility strains:

\[
\dot{M}^s = J \rho^s (\dot{n}^s + n^s \nabla \cdot \mathbf{v} - n^s \dot{\epsilon}_s^v) = 0
\]

\[
\dot{M}^f = \rho^f (\dot{n}^f + n^f \nabla \cdot \mathbf{v} - n^f \dot{\epsilon}_f^v) = -\nabla \cdot (\rho^f \mathbf{v}^d)
\]

(3.29)

where $\mathbf{v}^d = n^f \mathbf{v}^r$ was introduced as the Darcian velocity. Rewriting once again one obtains

\[
\dot{n}^f + n^f \nabla \cdot \mathbf{v} - n^f \dot{\epsilon}_f^v = -\frac{1}{\rho^f} \nabla \cdot (\rho^f \mathbf{v}^d)
\]

\[
\dot{n}^s + n^s \nabla \cdot \mathbf{v} - n^s \dot{\epsilon}_s^v = 0
\]

(3.30)

Combination with due consideration to the saturation constraint, i.e. $n^f + n^s = 1$; $\dot{n}^f + \dot{n}^s = 0$, leads to

\[
\dot{n}^f + \dot{n}^s + n^s \nabla \cdot \mathbf{v} + n^f \nabla \cdot \mathbf{v} - n^s \dot{\epsilon}_s^v - n^f \dot{\epsilon}_f^v = \nabla \cdot \mathbf{v} - n^s \dot{\epsilon}_s^v - n^f \dot{\epsilon}_f^v = -\frac{1}{\rho^f} \nabla \cdot (\rho^f \mathbf{v}^d)
\]

(3.31)

where we introduced the Darcian velocity defined as $\mathbf{v}^d = n^f \mathbf{v}^r$. We emphasize that the issue of “compressibility” relates to the changes of the densities $\rho^s$, $\rho^f$ following a solid particle $\varphi[X,t]$. This means, in particular, that $\rho^f = \rho^f[\varphi[X],t]$ at the assessment of fluid compressibility. The reason is that the initial density $\rho^f_0$ relates to $\mathcal{B}_0$ (and not $\mathcal{B}'_0$). In this context, we conclude that $\epsilon^s_0[x,t]:=0$ and $\epsilon^f_0[x,t]:=0$ corresponds the important situation of incompressible solid material and incompressible fluid phase materials, respectively. However, we may indeed have the situation that e.g. $\epsilon^s_v[x,t] \neq 0$ and $\epsilon^f_v[x,t] := 0$, corresponding a compressible solid phase material.
3.3 Balance of momentum

In this section the balance of momentum is addressed. The outline of the developments is defined by the theory questions:

1. Formulate the principle of momentum balance in total format. Formulate also the contribution of momentum from the different phases. Consider, in particular, the conservation of mass in the formulation.

2. Express the principle of momentum balance with respect to the individual phases. Focus on the solid phase reference configuration via the introduction of the relative velocity. In this, development express the localized equations of equilibrium for the different phases. Formulate your interpretation of the local interaction forces.

3.3.1 Total format

The linear momentum balance of the solid occupying the region $\mathcal{B}$ as in Fig. 3.2 may be written in terms of the “change” in total momentum $\mathcal{P}$ in $\mathcal{B}$, and the externally and internally applied forces $\mathcal{F}_{\text{ext}}$ and $\mathcal{F}_{\text{int}}$. The linear momentum balance relation is specified (as usual) as

$$\frac{D\mathcal{P}}{Dt} = \mathcal{F}_{\text{ext}} + \mathcal{F}_{\text{int}}$$

where the (total) forces of the mixture solid are defined as

$$\mathcal{F}_{\text{ext}} = \int_{\partial\mathcal{B}} \mathbf{t}_d \Gamma = \int_{\mathcal{B}} \mathbf{\bar{\sigma}} \cdot \nabla dv , \quad \mathcal{F}_{\text{int}} = \int_{\mathcal{B}} (\hat{\rho} \mathbf{g} + \hat{\rho}' \mathbf{g}) dv$$

Please, carefully note that $\mathbf{t}_d$ is the total traction vector acting along the external boundary $\partial\mathcal{B}$ and $\mathbf{g}$ is the gravity. The traction vector is related to the total stress tensor $\mathbf{\bar{\sigma}} = \mathbf{\sigma}^s + \mathbf{\sigma}^f$ via the outward normal vector $\mathbf{n}$ as

$$\mathbf{t}_d = \mathbf{\bar{\sigma}} \cdot \mathbf{n}$$

where $\mathbf{\sigma}^s$ and $\mathbf{\sigma}^f$ are the solid and fluid partial stresses, respectively.

In addition to linear momentum balance, angular momentum balance should be considered. However, if we restrict to the ordinary non-polar continuum representation the main result from this consideration is that the total stress (and also the partial stresses) is symmetric, i.e.

$$\mathbf{\bar{\sigma}} = \mathbf{\sigma}^t \Rightarrow \mathbf{\sigma}^s = (\mathbf{\sigma}^s)^t, \mathbf{\sigma}^f = (\mathbf{\sigma}^f)^t$$
In view of the fact that the momentum of our solid component, in the present context of a two phase material, consists of contributions from the individual phases let us consider the detailed formulation of the momentum change \( \frac{D \mathbf{P}}{Dt} \). To this end, we note in view of the mass balance relation (3.23) for the individual phases that the total change of momentum can be written as

\[
\frac{D \mathbf{P}}{Dt} := \frac{D^s \mathbf{P}^s}{Dt} + \frac{D^f \mathbf{P}^f}{Dt}
\]

where

\[
\frac{D^s \mathbf{P}^s}{Dt} = \frac{D^s}{Dt} \int_{B_0} M^s \mathbf{v} dV , \quad \frac{D^f \mathbf{P}^f}{Dt} = \frac{D^f}{Dt} \int_{B_0^f} J^f \dot{\rho}^f \mathbf{v}^f dV
\]

Note the solid phase in the mass balance relation (3.23) and its implication in the solid momentum conservation, i.e.

\[
\frac{D^s \mathbf{P}^s}{Dt} = \int_{B_0} (M^s \dot{\mathbf{v}} + \dot{M}^s \mathbf{v}) dV = \int_{B_0} (M^s \dot{\mathbf{v}} + J g^s \mathbf{v}) dV
\]
Likewise, for the fluid phase we obtain the detailed push-forward – pull-back consideration

\[ \frac{Df}{Dt} = \int_{Bf} \left( J^f \rho^f \frac{Df}{Dt} v^f + \frac{Df}{Dt} J^f \rho^f v^f \right) dV = \int_{Bf} \left( J^f \rho^f \frac{Df}{Dt} v^f + J^f g^f v^f \right) dV \]

\[ \int_{B} \left( \hat{\rho} \frac{Df}{Dt} v^f + g^f v^f \right) dv \]

where the last equality expresses the momentum change with respect to the solid phase material reference configuration. In addition, let us also express the fluid acceleration \( \frac{Df}{Dt} v^f \) in terms of the relative velocity \( v^f = v^r + v \), whereby the fluid acceleration can be related to the solid phase material via a convective term involving the relative \( v^r \). This is formulated via the parametrization \( v^f [\varphi^f [X^f], t] \) as:

\[ \frac{Df}{Dt} v^f = \frac{\partial v^f}{\partial t} + l^f \cdot v^f = \frac{\partial v^f}{\partial t} + l^f \cdot v + l^f \cdot v^r = \dot{v}^f + l^f \cdot v^r \]

where \( l^f = v^f \otimes \nabla \) is the spatial velocity gradient with respect to the fluid motion. Hence, the momentum change may be written as

\[ \frac{D^s P^s}{Dt} = \int_{B^s} (M^s \dot{v}^s + J^s g^s v) dV = \int_{B^s} (\hat{\rho} \dot{v} + g^s v) dv \]

\[ \frac{Df}{Dt} = \int_{Bf} \left( M^f \dot{v}^f + (v^f \otimes \nabla) \cdot v^r + J^f g^f v^f \right) dV = \int_{B} \left( \hat{\rho} \dot{v}^f + (\dot{v}^f + l^f \cdot v^r) + g^f v^f \right) dv \]

In view of the relations (3.32), (3.33), (3.41) and (3.42), we may finally localize the momentum balance relation for the two phase mixture as

\[ \frac{DP}{Dt} = F_{\text{ext}} + F_{\text{int}} \iff \sigma \cdot \nabla + \hat{\rho} g = \hat{\rho} \dot{v}^s + \hat{\rho} (\dot{v}^f + l^f \cdot v^r) + g^f v^r \forall x \in B \]

where \( \hat{\rho} = \hat{\rho}^s + \hat{\rho}^g \) and we made use of the fact that \( g^s + g^f = 0 \) and \( v^r = v^f - v \).

3.3.2 Individual phases and transfer of momentum change between phases

As alluded to in the previous sub-section we establish the resulting forces and momentum changes in terms of contributions from the individual phases, i.e. we have that

\[ \frac{DP}{Dt} = \frac{D^s P^s}{Dt} + \frac{Df}{Dt} P^f, \quad F_{\text{ext}} = F_{\text{ext}}^s + F_{\text{ext}}^f, \quad F_{\text{int}} = F_{\text{int}}^s + F_{\text{int}}^f \]
3.3. BALANCE OF MOMENTUM

where

\( \mathcal{F}_{\text{ext}}^s = \int_B \sigma^s \cdot \nabla dv \), \( \mathcal{F}_{\text{ext}}^f = \int_B \sigma^f \cdot \nabla dv \), \( \mathcal{F}_{\text{int}}^s = \int_B \rho^s g dv \), \( \mathcal{F}_{\text{int}}^f = \int_B \rho^f g dv \)

Hence, we are led to subdivide the momentum balance relation (3.44) as

\[
\begin{align*}
\frac{D^s P^s}{Dt} &= \mathcal{F}_{\text{int}}^s + \mathcal{F}_{\text{ext}}^s + \mathcal{H}^s \\
\frac{D^f P^f}{Dt} &= \mathcal{F}_{\text{int}}^f + \mathcal{F}_{\text{ext}}^f + \mathcal{H}^f
\end{align*}
\]

where \( \mathcal{H}^s \) and \( \mathcal{H}^f \) are interaction forces due to drag interaction between the phases. These are defined in terms of the local interaction forces \( h^s \) and \( h^f \)

\[
\mathcal{H}^s = \int_B h^s dv \ , \ \mathcal{H}^f = \int_B h^f dv
\]

In addition, it is assumed that the total effect of the interaction forces (or internal rate of momentum supply) will result in no change of the total momentum, i.e. we have that

\[
\mathcal{H}^s + \mathcal{H}^f := 0 \Rightarrow h^s + h^f = 0
\]

Hence, we may localize the relation (3.46) along with (3.41) and (3.42) as

\[
\begin{align*}
\sigma^s \cdot \nabla + \dot{\rho}^s g + h^s &= \dot{\rho}^s \dot{v} + g^s v \\
\sigma^f \cdot \nabla + \dot{\rho}^f g + h^f &= \dot{\rho}^f (\dot{v}^f + l^f \cdot v^f) + g^f v^f
\end{align*}
\]

Note that the summation of these relation indeed corresponds to the total form of the momentum balance specified in eq. (3.43).
3.4 Conservation of energy

In this section the conservation of energy is addressed with due consideration to the two involved phases. The developments are defined by the theory questions:

1. Establish the principle of energy conservation of the mixture material. Discuss the involved “elements” and formulate the internal energy and kinetic energy with respect to the solid reference configuration using the relative velocity.

2. “Work out” the mechanical work rate of the mixture solid.

3. Derive the energy equation using the assumption about an ideal viscous fluid. In this context, define effective stress of Terzaghi. Discuss the interpretation of the Terzaghi stress.

3.4.1 Total formulation

Let us first establish the principle of energy conservation (=first law of thermodynamics) written as the balance relation applied to the mixture of solid and fluid phases as

\[
\frac{DE}{Dt} + \frac{DK}{Dt} = W + Q
\]

where \( E = E^s + E^f \) is the total internal energy and \( K = K^s + K^f \) is the total kinetic energy of the mixture solid and the total material velocity with respect to the mixture material is defined as \( \dot{\mathbf{x}}^m := \frac{D\mathbf{x}^s}{Dt} + \frac{D\mathbf{x}^f}{Dt} \). Moreover, \( W \) is the mechanical work rate of the solid and \( Q \) is the heat supply to the solid. In the following we shall not disregard \( Q \) in the energy balance although we will confine ourselves to isothermal conditions subsequently. The different “elements” involved in the energy balance are depicted in Fig. 3.3 below.

3.4.2 Formulation in contributions from individual phases

The individual contributions \( E^\alpha \) and \( K^\alpha \) to the total internal and kinetic energies \( E, K \) are defined as

\[
E^s = \int_B \dot{\rho}^s e^s dV = \int_{B_0} M^s e^s dV , \quad \dot{E}^f = \int_B \dot{\rho}^f e^f dV = \int_{B_0} J^f \dot{\rho}^f e^f dV
\]

\[
K^s = \int_B \dot{\rho}^s k^s dV = \int_{B_0} M^s k^s dV , \quad \dot{K}^f = \int_B \dot{\rho}^f k^f dV = \int_{B_0} J^f \dot{\rho}^f k^f dV
\]
3.4. CONSERVATION OF ENERGY

Figure 3.3: Quantities involved in the formulation of the principle of energy conservation.

where $e^s$ is the internal energy density (per unit mass) pertinent to the solid phase and $e^f$ is the internal energy density pertinent to the fluid phase. Moreover, we introduced $k^s = \frac{1}{2}v \cdot v$ and $k^f = \frac{1}{2}v^f \cdot v^f$. The material time derivatives of the involved contributions thus becomes

\begin{align}
\frac{D^s e^s}{Dt} &= \frac{D^s}{Dt} \int_{B_0} M^s \dot{e}^s dV = \int_{B_0} (M^s \dot{e}^s + Jg^s e^s) dV \\
\frac{D^f e^f}{Dt} &= \frac{D^f}{Dt} \int_{B_0} f^f \dot{\rho}^f e^f dV = \int_{B} \left( \dot{\rho}^f \frac{D^f e^f}{Dt} + g^f e^f \right) dV = \int_{B_0} \left( M^f \frac{D^f e^f}{Dt} + Jg^f e^f \right) dV
\end{align}

where the mass balance relations in (3.23) were used.

Let us next reformulate the material time derivatives relative to the solid reference
configuration using the relative velocity \( v^r \). To this end, it is first noted that

\[
\frac{D^s e^s}{Dt} = \dot{e}^s, \quad \frac{D^f e^f}{Dt} = \frac{\partial e^f}{\partial t} + (\nabla e^f) \cdot v + (\nabla e^f) \cdot v^r = \dot{e}^f + (\nabla e^f) \cdot v^r
\]

which leads to

\[
\frac{DE}{Dt} + \frac{DK}{Dt} = \int_B \left( \hat{\rho}^s \dot{\varepsilon}^s + \hat{\rho}^f \dot{\varepsilon}^f + \hat{\rho}^f \left( \nabla e^f \right) \cdot v^r + g^s e^s + g^f e^f \right) dv + \int_B \left( \hat{\rho}^s v \cdot \dot{v} + \hat{\rho}^f v^f \cdot \frac{D^f v^f}{Dt} + g^s k^s + g^f k^f \right) dv = \]

\[
\int_B \left( \dot{\varepsilon} + \hat{\rho}^f \left( \nabla e^f \right) \cdot v^r \right) dv + \int_B \left( \hat{\rho}^s v \cdot \dot{v} + \hat{\rho}^f v^f \cdot \frac{D^f v^f}{Dt} \right) dv + \int_B g^f \left( e^f + k^f - (e^s + k^s) \right) dv
\]

In the last equality we introduced the (solid) material change of internal energy of the mixture defined as

\[
\dot{\varepsilon} = \hat{\rho}^s \dot{\varepsilon}^s + \hat{\rho}^f \dot{\varepsilon}^f
\]

### 3.4.3 The mechanical work rate and heat supply to the mixture solid

In view of Fig. 3.3 we may “work out” the mechanical work rate produced by the gravity forces in \( B \) and the forces acting on the external boundary \( \Gamma \). This is formulated as

\[
W = \int_B \left( \hat{\rho}^s g \cdot v + \hat{\rho}^f g \cdot v^f \right) dv + \int_{\Gamma} \left( v \cdot (\sigma^s \cdot n) + v^f \cdot (\sigma^f \cdot n) \right) d\Gamma \]

where the last expression was obtained using the divergence theorem “DIV” (and along with some additional derivations). Combination of this last relationship with the equilibrium relations of the phases in eq. (3.49) yields the work rates of our mixture continuum as

\[
W = \int_B \left( \hat{\rho}^s \dot{v} \cdot v + \hat{\rho}^f \frac{D^f v^f}{Dt} \cdot v^f \right) dv + \int_B \left( \sigma^s : l + \sigma^f : l^f - h^f \cdot v^r \right) dv + \int_B \left( g^s v \cdot v + g^f v^f \cdot v^f \right) dv
\]
3.4. CONSERVATION OF ENERGY

Let us consider next the heat supply $Q$, cf. Fig. 3.3, to the solid formulated as

\begin{equation}
Q = - \int_{\Gamma} \mathbf{n} \cdot \mathbf{h} d\Gamma = \int_{\Omega} \nabla \cdot \mathbf{h} dv = - \int_{\Omega} q dv
\end{equation}

where the last equality was obtained by using the divergence theorem. We note that $\mathbf{h}$ is the thermal flux vector that transports heat energy from the mixture solid, and $q = \nabla \cdot \mathbf{h}$ is the divergence of thermal flow. In particular, we have that $q > 0$ corresponding to production of energy at the material point.

### 3.4.4 Energy equation in localized format

Hence the balance of energy stated in (3.50) along with (3.56), (3.59) and (3.60) can now be formulated as

\begin{align}
\int_{\Omega} \dot{e} dv + \int_{\Omega} \ddot{\rho} \left( e' \nabla \right) \cdot \mathbf{v} r dv + \int_{\Omega} g^f \left( e^f + k^f - (e^s + k^s) \right) dv = \\
\int_{\Omega} \left( \sigma^s : \mathbf{l} + \sigma^f : \mathbf{l}^f - \mathbf{h}^f \cdot \mathbf{v} r \right) dv + \int_{\Omega} g^f \left( \mathbf{v} f \cdot \mathbf{v} f - \mathbf{v} \cdot \mathbf{v} \right) dv - \int_{\Omega} q dv
\end{align}

(3.61)

In view of (3.61), let us immediately specify the localized format of the energy equation (while noting that $k^s = \frac{1}{2} \mathbf{v} \cdot \mathbf{v}$ and $k^f = \frac{1}{2} \mathbf{v} f \cdot \mathbf{v} f$) as

\begin{align}
\dot{\epsilon} + \ddot{\rho} \left( \nabla e^f \right) \cdot \mathbf{v} r + g^f \left( e^f - e^s \right) = \\
\sigma^s : \mathbf{l} + \sigma^f : \mathbf{l}^f - \mathbf{h}^f \cdot \mathbf{v} r + g^f \left( k^f - k^s \right) - q
\end{align}

(3.62)

Once again, let us represent the fluid phase term $\sigma^f : \mathbf{l}^f$ in (3.62) to the motion of the solid phase. Hence, we rewrite the term $\sigma^f : \mathbf{l}^f$ as

\begin{align}
\sigma^f : \mathbf{l}^f = & \sigma^f : \left( \mathbf{l} + \mathbf{l}^r \right) = \left\{ \sigma^f : \mathbf{l}^r = \nabla \cdot \left( \mathbf{v} r \cdot \sigma^f \right) - \mathbf{v} r \cdot \nabla \cdot \sigma^f \right\} = \\
& \sigma^f : \mathbf{l} + \nabla \cdot \left( \mathbf{v} r \cdot \sigma^f \right) - \mathbf{v} r \cdot \nabla \cdot \sigma^f = \\
& \sigma^f : \mathbf{l} + \nabla \cdot \left( \mathbf{v} r \cdot \sigma^f \right) + \mathbf{v} r \cdot \left( \mathbf{h}^f + \ddot{\rho} f \left( g - \frac{D^f v^f}{Dt} \right) - g^f \mathbf{v} f \right)
\end{align}

(3.63)

where the equilibrium relation (3.49b) was used once again. Hence, the relation (3.62) is re-established in terms of the total stress $\sigma$ of the mixture as

\begin{align}
\dot{\epsilon} + q + g^f \left( e^f - e^s \right) = \\
\left( \sigma^s + \sigma^f \right) : \mathbf{l} + \nabla \cdot \left( \mathbf{v} r \cdot \sigma^f \right) + \ddot{\rho} f \mathbf{v} r \cdot \left( g - \frac{D^f v^f}{Dt} \right) - \rho f \mathbf{v} r \cdot \left( \nabla e^f \right) - g^f \frac{1}{2} \mathbf{v} r \cdot \mathbf{v} r = \\
\sigma : \mathbf{l} + \nabla \cdot \left( \mathbf{v} r \cdot \sigma^f \right) + \ddot{\rho} f \mathbf{v} r \cdot \left( g - \frac{D^f v^f}{Dt} - \nabla e^f \right) - g^f \frac{1}{2} \mathbf{v} r \cdot \mathbf{v} r
\end{align}

(3.64)
3.4.5 Assumption about ideal viscous fluid and the effective stress of Terzaghi

Using the assumption about the ideal viscous fluid, the fluid stress may be represented as

\[
\sigma^f = n^f s^f - n^f p^f 1
\]

where \( s^f \) is the intrinsic (normally viscous) portion of the fluid stress and \( p^f \) is the intrinsic (non-viscous) fluid pressure. We note already at this point that \( s^f \) is completely deviatoric, and the non-viscous fluid pressure is equal to the total fluid pressure. Nevertheless, we formulate the term \( \nabla \cdot (v^r \cdot \sigma^f) \) of (3.64) as

\[
\nabla \cdot (v^r \cdot \sigma^f) = \nabla \cdot (v^d \cdot s^f) - \nabla \cdot \left( \frac{\rho^f p^f}{\rho^d} v^r \right) = \nabla \cdot (v^d \cdot s^f) - \frac{\rho^f}{\rho^d} \nabla \cdot (\rho^f v^d) - \rho^f v^d \nabla \cdot \left( \frac{p^f}{\rho^f} \right)
\]

where the introduction of the Darcian velocity \( v^d = n^f v^r \) is noteworthy. As a result the energy balance in (3.66) becomes

\[
\dot{\varepsilon} + g^f (e^f - e^s) = \sigma : l + \nabla \cdot (v^d \cdot s^f) - \frac{\rho^f}{\rho^d} \nabla \cdot (\rho^f v^d) +
\]

\[
\rho^f v^d \cdot \left( g - \frac{D^f v^f}{Dt} - \nabla e^f - \nabla \cdot \left( \frac{p^f}{\rho^f} \right) \right) - g^f \frac{1}{2} v^r \cdot v^r
\]

Upon combining the term \( -\frac{\rho^f}{\rho^d} \nabla \cdot (\rho^f v^d) \) in (3.67) with the balance of mass of the mixture material in eq. (3.31) we obtain

\[
-\frac{\rho^f}{\rho^d} \nabla \cdot (\rho^f v^d) = \rho \nabla \cdot v - n^s p^s \dot{e}^s_v - n^f p^f \dot{e}^f_v
\]

leading to

\[
\dot{\varepsilon} + q + g^f (e^f - e^s) = \sigma : l + \nabla \cdot (v^d \cdot s^f) - n^s p^s \dot{e}^s_v - n^f p^f \dot{e}^f_v +
\]

\[
\rho^f v^d \cdot \left( g - \frac{D^f v^f}{Dt} - \nabla e^f - \nabla \cdot \left( \frac{p^f}{\rho^f} \right) \right) - g^f \frac{1}{2} v^r \cdot v^r
\]

with the effective stress \( \sigma \) of Terzaghi defined as

\[
\sigma = \sigma^f + p^f 1
\]

In order to consider the influence of the viscous contribution to the fluid stress we develop the term \( \nabla \cdot (v^r \cdot s^f) \) in its indices as

\[
\nabla_k ((v^d)_j s^f)_{jk} = (s^f)_{kj} (v^d)_{jk} + ((v^d)_j \nabla_k (s^f))_{jk} = s^f : l^d + v^d \cdot (\nabla \cdot s^f)
\]
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which leads to

\[ \dot{t} + g + g^f (e^f - e^s) = (\sigma - n^f s^f) : l + n^f s^f : l^f - n^s p^s - n^f p^f + \]

\[ \rho^f v^d \left( g - D^f v^f \right) - \nabla e^f - \nabla \left( \frac{p}{\rho^f} \right) + \frac{1}{\rho^f} \nabla \cdot s^f - g^f \frac{1}{2} v^r \cdot v^r \]

(3.72)

We thus conclude that the effective stress \( \sigma - n^f s^f \) (and not the partial stress of the solid) that is "felt" by the continuum. It should be emphasized, however, that it is normally assumed that the viscous portion \( s^f \) of the fluid stress may be neglected, i.e. \( s^f \approx 0 \).

Hence, for practical purposes it is desirable to consider the effective stress as the one that controls the "strength" of a porous medium. In soil mechanics, for example, it is commonly accepted that the strength of soil is dependent on the Terzaghi effective stress. Thereby the "principle" of effective stress of Terzaghi states that the strength of the skeleton is an intrinsic property and it does not depend on the fluid pressure. One common engineering argument for its particular expression, e.g. Zienkiewicz et al. [44], is that instead of taking a typical cut in the porous medium it is more rational to consider "sections" determining the pore water effect through arbitrary surfaces with minimum contact points of the solid skeleton. It has also been argued that the compressibility of the solid material plays a role in the statement of the effective stress. Much thought has been addressed to this hypothesis, and we may refer to de Boer and Ehlers [15], Lade and de Boer [32] and Bluhm and de Boer [12] for some contributions on this topic. From the above discussion it follows that the constitutive description of the solid matrix should involve the effective stress rather than the partial stress of the solid.
3.5 Entropy inequality

In this section the entropy inequality pertinent to a two-phase continuum is outlined. The developments of this section are defined by the theory questions:

1. Establish the total entropy and the second law of thermodynamics in the context of a two-phase mixture material. On the basis of the involved material derivatives and principle of mass conservation, express the entropy inequality in terms of the solid reference configuration.

2. Define and discuss the suitable Legendre transformation in internal energy, Helmholtz free energy, temperature and entropy. Represent and localize the entropy inequality of the mixture material in terms of these quantities.

3. On the basis of the localized entropy for the mixture material, discuss the different components of the dissipation inequality and relate them to dissipative mechanics of the mixture material.

3.5.1 Formulation of entropy inequality

Let us next consider the second law of thermodynamics formulated in terms of the total entropy $S$ written as

$$S = \int_B (\hat{\rho}^s s^s + \hat{\rho}^f s^f) \, dv$$

where $s^s$ and $s^f$ are the local entropies per unit mass of the solid and fluid phases, respectively. As to the temperature, it is assumed that it is constant, i.e. it is stationary, and in common for both phases, i.e. we have that $\theta = \theta^s = \theta^f$.

The second law of thermodynamics (which is sometimes also named the “Clausius Duhem's Inequality” (CDI), or simply the “entropy inequality”) is now stated as

$$\frac{DS}{Dt} - Q_\theta \geq 0$$

where $\frac{DS}{Dt}$ is the total time derivative of our two-phase porous material as we have discussed extensively (so far) during the course. Moreover, $Q_\theta$ is the net heat thermal supply/per temperature unit defined as

$$Q_\theta = -\int_{\Gamma} \frac{n \cdot h}{\theta} d\Gamma_{DIV} - \int_B \nabla \cdot \left( \frac{h}{\theta} \right) \, dv = -\int_B \frac{1}{\theta}(\nabla \cdot h - h \cdot \nabla \theta) \, dv = -\int_B \frac{1}{\theta}(q - h \cdot \nabla \theta) \, dv$$

where $\theta > 0$ is the absolute temperature of the medium.
3.5. ENTROPY INEQUALITY

With due consideration to the balance of mass stated in eq. (3.23) (involving the mass transfer factor \( g^f \)), we develop the total material derivative of the total entropy stated in eq. (3.74). Thereby, the total material change of the entropy is obtained as

\[
\frac{DS}{Dt} = \int_B \left( \dot{\rho}^s s^s + \dot{\rho}^f s^f + g^f (s^f - s^s) \right) dv
\]

In particular, the material time derivative of the fluid is represented in terms of that of the solid phase (in the usual way), i.e.:

\[
\begin{align*}
\frac{D^s s^s}{Dt} &= \dot{s}^s \\
\frac{D^f s^f}{Dt} &= \frac{\partial s^f}{\partial t} + (\nabla s^f) \cdot v + (\nabla s^f) \cdot v^r = \dot{s}^f + (\nabla s^f) \cdot v^r
\end{align*}
\]

which yields the material change of the total entropy in (3.76) as

\[
\frac{DS}{Dt} = \int_B \left( \dot{\rho}^s s^s + \dot{\rho}^f s^f + \dot{\rho}^f (\nabla s^f) \cdot v^r + g^f (s^f - s^s) \right) dv =
\]

\[
\int_B \left( \dot{s} + \dot{\rho}^f (\nabla s^f) \cdot v^r + g^f (s^f - s^s) \right) dv
\]

where \( \dot{s} \) is the saturated entropy change defined as

\[
\dot{s} = \dot{\rho}^s s^s + \dot{\rho}^f s^f
\]

3.5.2 Legendre transformation between internal energy, free energy, entropy and temperature

We relate generically the internal energy \( e \), the free energy \( \psi \) and the entropy \( s \) to each other via the Legendre transformation

\[
e[A, s] = \psi[A, \theta] + s \theta
\]

where \( \psi \) is the Helmholtz free energy (or simply the free energy) representing the stored reversible energy as a function of the internal variables \( A \). We emphasize that the set \( \{s, \theta, A\} \) (where \( \theta \) is the temperature) represents the independent variables in eq. (3.80). As a result the linearized Legendre transformation reads

\[
\begin{align*}
\frac{\partial e}{\partial A} \dot{A} + \frac{\partial e}{\partial s} \dot{s} &= \frac{\partial \psi}{\partial A} A + \frac{\partial \psi}{\partial \theta} \dot{\theta} + s \dot{\theta} + \dot{s} \theta \\
\left( \frac{\partial e}{\partial A} - \frac{\partial \psi}{\partial A} \right) \dot{A} + \left( \frac{\partial e}{\partial s} - \theta \right) \dot{s} &= \left( \frac{\partial \psi}{\partial \theta} + s \right) \dot{\theta}
\end{align*}
\]
corresponding to the conditions

\[
\frac{\partial e}{\partial A} = \frac{\partial \psi}{\partial A}, \quad \theta = \frac{\partial e}{\partial s}, \quad s = -\frac{\partial \psi}{\partial \theta}
\]

whereby a clear interpretation of the entropy is obtained, i.e. the “entropy” is the sensitivity of the free energy with respect to the temperature.

In the following, let us apply the same transformation with respect to the individual phases. Hence, we introduce the relationships for the solid phase

\[
\theta = e - \psi \Rightarrow \dot{\theta} = \dot{e} - \dot{\psi}
\]

(3.83)

where \(\psi\) is the free energy of the solid phase. However, in the following we shall consider the isothermal condition leading to

(3.84)

\[
\dot{\theta} = \dot{e} - \dot{\psi}
\]

Likewise, for the fluid phase we introduce the Legendre transformation

\[
\theta = e - \psi \Rightarrow \dot{\theta} = \dot{e} - \dot{\psi} + (\nabla s) \cdot \mathbf{v}
\]

(3.85)

where \(\psi\) is the free energy of the fluid phase.

In addition, let us also consider the saturated entropy \(\dot{\hat{s}}\) in (3.79). In view of (3.84) and (3.85) one obtains

(3.86)

\[
\dot{\hat{s}} = \dot{\hat{e}} - \dot{\hat{\psi}}
\]

3.5.3 The entropy inequality - Localization

In view of the relations (3.74), (3.75) and (3.78), the entropy inequality now reads

(3.87)

\[
\frac{DS}{Dt} = \int_B \left( \dot{\hat{s}} + \dot{\hat{\psi}} (\nabla (s^f)) \cdot \mathbf{v} + g^f (s^f - s^s) \right) dv + \int_B \frac{1}{\theta} (q - h \cdot \nabla \theta) dv \geq 0
\]

It appears that the localized form of the entropy inequality may be written as

(3.88)

\[
D_{\text{mech}} + D_{\text{ther}} \geq 0
\]
where we require that the thermal and mechanical portions of the inequality are assumed to the satisfied independently as

\begin{equation}
D^\text{ther} = -h \cdot \nabla \theta \geq 0 \Rightarrow h = -K^\text{ther} \cdot \nabla \theta \Rightarrow D^\text{ther} \geq 0
\end{equation}

\begin{equation}
D^\text{mech} = \dot{s} + \dot{\rho} f \left( \nabla (\theta s^f) - s^f \nabla \theta \right) \cdot \mathbf{v}^r + g^f (\theta s^f - \theta s^s) + q \geq 0
\end{equation}

In (3.90), it was used that \( \theta \nabla s^f = \nabla (\theta s^f) - s^f \nabla \theta \), and in (3.89) the thermal part of the entropy inequality is immediately satisfied via the introduction of Fourier’s law of heat conduction, where \( K^\text{ther} \) is the second order positive definite thermal conductivity tensor.

As to the mechanical portion, we develop \( D^\text{mech} \) in (3.90) using the Legendre transformations in the sequel (3.86-3.80). As a result we now obtain

\begin{equation}
D^\text{mech} = \dot{\varepsilon} - \dot{\psi} + \dot{\rho} f \left( \nabla (\varepsilon^f - \psi^f) - s^f \nabla \theta \right) \cdot \mathbf{v}^r + g^f (\varepsilon^f - \psi^f - (\varepsilon^s - \psi^s)) + q \geq 0
\end{equation}

Combination with the energy equation (3.72) yields:

\begin{equation}
\begin{align*}
\left( \sigma - n^f s^f \right) : l &- n^s p \dot{\varepsilon}^s - \dot{\rho}^s \dot{\psi}^s + \\
n^f s^f : l^f + \\
- n^f \dot{\rho}^f \dot{\psi}^f + \\
\rho^f \mathbf{v}^d : \left( g - \frac{D^f \mathbf{v}^f}{Dt} + \frac{1}{\rho^f} \nabla \cdot s^f - \nabla \psi^f - \nabla \cdot \left( \frac{p}{\rho^f} \right) - s^f \nabla \theta - \frac{1}{2 \dot{\rho}^f} \mathbf{v}^r \right) + \\
g^f (-\psi^f + \psi^s) \geq 0
\end{align*}
\end{equation}

whereby the final result in fact may be interpreted in terms of a number of independent phenomenological mechanisms of the mixture material. In view the appearance of the different terms in (3.92). This is formulated as

\begin{equation}
D = D^s + D^\text{visc} + D^i + D^e \geq 0
\end{equation}

where \( D^s \geq 0 \) is the dissipation produced by the (homogenized) solid phase material considered as an independent process of the mixture. The term \( D^\text{visc} \geq 0 \) represents the dissipation developed by the viscous portion of the fluid stress \( \sigma^f \). The term \( D^\text{visc} \) represents dissipation in the non-viscous stress response of the fluid. It is assumed that this dissipation can be neglected, i.e. \( D^\text{visc} \equiv 0 \). The term \( D^i \geq 0 \) represents dissipation induced by “drag”-interaction between the phases. Finally, the term \( D^e > 0 \) represents dissipation motivated by mass transfer between the phases.

Motivated by the entropy inequality (3.92) the different components of the total dissipation are defined as

\begin{equation}
D^s = \sigma : l - n^s p \dot{\varepsilon}^s - n^s \dot{\rho}^s \dot{\psi}^s \geq 0
\end{equation}
(3.95) \[ D^{vf} = n^f s^f : \mathbf{U}^f \geq 0 \]

(3.96) \[ D^{nvf} = -n^f \left( p \dot{e}_v^f + \rho^f \dot{\psi}^f \right) = 0 \]

\[ D^i = -h_e^f \cdot \mathbf{v}^d \geq 0 \text{ with} \]

(3.97) \[ h_e^f = -\rho^f \left( g - \frac{D^f v^f}{D t} + \frac{1}{\rho^f} \nabla \cdot s^f - \nabla \left( \frac{p}{\rho^f} \right) - \nabla \psi^f - s^f \nabla \theta - \frac{1}{2} g^f \mathbf{v}^r \right) \]

(3.98) \[ D^e = g^f (\psi^s - \psi^f) \geq 0 \Rightarrow g^f \geq 0, \psi^s - \psi^f \geq 0 \]

In (3.97), \( h_e^f \) is the effective drag force. Moreover, we introduced \( D^s \geq 0 \) as the dissipation produced by (homogenized) solid phase material, \( D^{vf} \geq 0 \) is the dissipation developed by viscous portion of the fluid stress \( \sigma^f \), \( D^i \geq 0 \) represents dissipation induced by "drag"-interaction between phases, and, finally, \( D^e > 0 \) is the dissipation motivated by mass transfer between the phases. In addition, from the assumption that the non-viscous portion of the fluid stress response is non-dissipative it immediately follows that the argument of \( \psi^f \) is the compressibility strain \( \epsilon_v^f \) defined by \( \psi^f = \psi^f [\epsilon_v^f] \) pertinent to a compressible fluid. It then follows from (3.96) that the fluid pressure can be evaluated directly from the compressibility \( \epsilon_v^f \) via the state \( \psi^f [\epsilon_v^f] \) as

(3.99) \[ p \dot{e}_v^f + \rho^f \frac{\partial \psi^f}{\partial \epsilon_v^f} \dot{\epsilon}_v^f = 0 \Rightarrow \boxed{p = -\rho^f \frac{\partial \psi^f}{\partial \epsilon_v^f}} \]

Alternatively, we may consider the constitutive relation in “compliance” form, in which case the fluid pressure \( p \) is the primary variable that produces the compressibility \( \epsilon_v^f \).

3.5.4 A note on the effective drag force

It is of significant interest to develop the effective drag force \( h_e^f \) in terms of \( h^f = -h^s \), which is the actual one appearing in the momentum balance of the fluid phase in (3.49b). To this end, we first note that the constitutive relation \( p = -\rho^f \frac{\partial \psi^f}{\partial \epsilon_v^f} \) yields the result

(3.100) \[ \nabla \left( \frac{p}{\rho^f} \right) = -\frac{1}{\rho^f} \nabla p + \left( \frac{1}{\rho^f} - \frac{\partial \rho^f}{\partial \epsilon_v^f} \frac{\partial \psi^f}{\partial \rho^f} \right) \nabla \rho^f = \]

\[ \{\epsilon_v^f = -\log \left[ \frac{\rho^f}{\rho_0} \right] \} = -\frac{1}{\rho^f} \nabla p + \left( \frac{1}{\rho^f} - \frac{\partial \psi^f}{\partial \epsilon_v^f} \rho^f \right) \nabla \rho^f = -\frac{1}{\rho^f} \nabla p \]

Hence, one obtains the effective drag force \( h_e^f \) in (3.97) stated in the form

(3.101) \[ -n^f h_e^f = n^f \nabla \cdot (s^f - p) + \rho^f \left( g - \frac{D^f v^f}{D t} \right) - n^f s^f \nabla \theta - \frac{1}{2} g^f \mathbf{v}^r \]
3.5. ENTROPY INEQUALITY

Let us next assume, firstly, isothermal conditions corresponding to \( \psi^f [c_e^f] \Rightarrow s^f = -\frac{\partial \psi^f}{\partial \theta} = 0 \), and, secondly, a non-erosive process \( g^f := 0 \) which gives

\[
(3.102) \quad - n^f h^f_e = n^f \nabla \cdot (s^f - p1) + \dot{\rho}^f \left( g - \frac{D^f v^f}{Dt} \right)
\]

To arrive at the relation between \( h^f_e \) and \( h^f \), let us reconsider the momentum balance relation (0b) of the fluid phase written as

\[
(3.103) \quad - h^f = \sigma^f \cdot \nabla + \dot{\rho}^f \left( g - \frac{D^f v^f}{Dt} \right) = n^f (s^f - p1) \cdot \nabla + \dot{\rho}^f \left( g - \frac{D^f v^f}{Dt} \right) + (s^f - p1) \cdot \nabla n^f - g^f v^f = - h^f_e + (s^f - p1) \cdot \nabla n^f
\]

Hence, the effective drag force is related to the actual one via the relation

\[
(3.104) \quad h^f = n^f h^f_e - (s^f - p1) \cdot \nabla n^f
\]

The momentum balance (3.49b) can thus be restated as

\[
(3.105) \quad \rho^f \frac{D^f v^f}{Dt} = \nabla \cdot s^f - \nabla p + h^f_e + \rho^f g
\]

which is the momentum balance of the fluid phase represented in intrinsic quantities. We note that in this relationship the effective drag force \( h^f_e \) plays the role of the “actual” drag force.
3.6 Constitutive relations

Guided by the conservation principles formulated for the two-phase mixture, we outline in this section constitutive relations for the different phase as well as for their interaction. The developments are defined by the theory questions:

1. Derive the state equation for the effective stress in the context of an hyperelastic material. Formulate also the transformation between our different stress measures. Carry out the derivation with the restriction to a non-erosive mixture material. What would happen if the material would have been erosive?

2. Motivated by the dissipation $D^i$ in eq. (3.92), formulate Darcy’s law as a constitutive representation of the drag interaction between the phases. Discuss under which circumstances we can write:
   \[ \mathbf{v}^d = -k \left( \nabla p - \rho f \left( g - \frac{D^f \mathbf{w}^f}{D t} \right) \right) \]
   where $k$ is the (isotropic) permeability constant.

3. Formulate the representation of the fluid phase as a gas phase. In this development, derive the fluid compressibility!

Guided by the dissipation inequality we outline, in this section, the constitutive relations of our two-phase continuum. We then consider in turn the different mechanisms outlined in the sequel (3.98). In the following, we shall assume that the mass production term $g^f = -g^s = 0$ pertinent to a non-erosive process.

3.6.1 Effective stress response

To arrive at the proper formulation of the effective stress response, let us formulate the dissipation produced by the solid phase for the entire component as

\[ D^s = \int_{B} D^s dv = \int_{B_0} JD^s dV \geq 0 \]

where the last expression was obtained after pull-back to the solid reference configuration. It may be noted that the integrand $JD^s$ in (3.106) may be rewritten in terms of the Kirchhoff stress $\tau = J \sigma$ (and the second Piola Kirchhoff stress $S$) as

\[ JD^s = \tau : l - n^s p \hat{c}^s_w - J n^s \rho \dot{\psi}^s = \frac{1}{2} S : \dot{\mathbf{C}} - J n^s p \hat{c}^s_w - \hat{\rho}_0 \dot{\psi}^s \]

where, in particular, the last equality was obtained from mass conservation due to the stationarity of $M^s = M_0^s = \hat{\rho}_0^s$. 
3.6. CONSTITUTIVE RELATIONS

Let us for simplicity assume hyper-elasticity “HE”, whereby we obtain the dependence \( \psi^s [C, \epsilon^s_v] \) in the free energy for the solid phase. We then have \( JD^s := 0 \) leading to

\[
\psi^s [C, \epsilon^s_v] = \frac{\partial \psi^s}{\partial C} : \dot{C} + \frac{\partial \psi^s}{\partial \epsilon^s_v} \dot{\epsilon}^s_v \Rightarrow
\]

\[
JD^s = \frac{1}{2} \left( S - 2\hat{\rho}_0 \frac{\partial \psi^s}{\partial C} \right) : \dot{C} - Jn^s \left( p + \rho^s \frac{\partial \psi^s}{\partial \epsilon^s_v} \right) \dot{\epsilon}^s_v \overset{\text{HE}}{=} 0
\]

We thus obtain the constitutive state equations at hyper-elasticity:

\[
S = 2\hat{\rho}_0 \frac{\partial \psi^s}{\partial C}
\]

\[
p = -\rho^s \frac{\partial \psi^s}{\partial \epsilon^s_v}
\]

where \( S = \overline{S} - JC^{-1}p \) is the consequent effective second Piola Kirchhoff stress due to the Terzahgi effective stress principle, and \( p \) is the intrinsic fluid pressure. Like in the formulation of the fluid compressibility in (3.99), we may alternatively consider the constitutive relation (3.110) in “compliance” form, whereby (again) the fluid pressure \( p \) drives the solid compressibility \( \epsilon^s_v \).

3.6.2 Solid-fluid interaction

Related to the constitutive relation for the densification of the gas, we consider the model for the drag interaction motivated by the dissipation \( D^i \) in (3.98) written as

\[
D^i = -h^f_e \cdot v^d \geq 0
\]

where the effective drag force \( h^f_e \) (or hydraulic gradient with negative sign) is chosen to ensure positive dissipation, cf. the representation of Fourier’s law of heat conduction in (3.89), according to the linear Darcy law

\[
v^d = -K \cdot h^f_e
\]

where \( K \) is the positive definite permeability tensor. To simplify the analysis let us restrict to isotropic permeability conditions using the scalar valued permeability parameter \( k \), whereby \( K = k1 \) and \( k > 0 \).

Experimental observations shows that the permeability \( k \) may be expressed in the effective gas-fluid viscosity \( \mu^f \) and the intrinsic permeability \( K^s \) of the solid material. This is written as

\[
- h^f_e = \frac{\mu^f}{K^s} v^d \Rightarrow k = \frac{K^s}{\mu^f}
\]
Neglecting viscous part of the fluid stress $s_f$ in (3.102), we obtain

$$v^d = -k \left( \nabla p - \rho_f \left( g - \frac{D_f v^f}{D_t} \right) \right)$$

To conclude the developments in this subsection, we remark that the special situation of “impermeable material” (or closed cells in a cellular material), is named the “undrained case”. In this case we have $v^d := 0$ and $D^i := 0$, corresponding to $K^s \to 0$.

### 3.6.3 Viscous fluid stress response

It appears that the deviatoric fluid stress response $s_f$ generates the dissipative portion $D^{\text{vf}} = n_f s_f : \mathbf{U} \geq 0$. To ensure positive dissipation, the interpretation of $s_f$ is then that the corresponding stress response is purely viscous and thereby completely dissipative. This is formulated as

$$s_f = 2 \mu_f I^{\text{dev}} : \mathbf{U} \Rightarrow D^{\text{vf}} = n_f s_f : \mathbf{U} = 2 \mu_f n_f \mathbf{U} : I^{\text{dev}} : \mathbf{U} \geq 0$$

where $\mu_f$ is the intrinsic fluid viscosity parameter and $I^{\text{dev}}$ is the forth order deviatoric projection operator.

### 3.6.4 Solid densification - incompressible solid phase

The intrinsic density of the solid phase is normally considered as a constant during the deformation process of the material. This corresponds to the usual assumption of incompressible solid phase material. Hence, we have $\epsilon^s_v := 0$ and

$$\epsilon^s_v = -\log \left[ \frac{\rho^s}{\rho_0^s} \right] := 0 \Rightarrow \rho^s = \rho_0^s$$

In addition, for hyper-elasticity we obtain the dependence $\psi^s \left[ \mathbf{C}, \epsilon^s_v \right] \to \psi^s \left[ \mathbf{C} \right]$ in the free energy for the solid phase.

### 3.6.5 Fluid densification - incompressible liquid fluid phase

The intrinsic density of the fluid phase considered as a liquid is also normally considered as a constant during the deformation process of the mixture material. This corresponds to the usual assumption of incompressible fluid phase material. Hence, we have $\epsilon^f_v := 0$ and

$$\epsilon^f_v = -\log \left[ \frac{\rho^f}{\rho_0^f} \right] := 0 \Rightarrow \rho^f = \rho_0^f$$
In addition, we obtain the dependence \( \psi^f[\epsilon^f_v] \to 0 \) in the free energy for the fluid phase.

### 3.6.6 Fluid phase considered as gas phase

In some applications it is of interest to model the fluid phase considered as a compressible gas concerning the non-viscous pressure response, e.g. gas-filled foam materials subjected to large rapid deformations. To this end, the gas-pressure response is modeled using the ideal gas law (or the Boyle–Mariotte’s law) written as

\[
\rho^f = \frac{m^g}{R \theta^f} \rho^i
\]

where \( R \) is the universal gas constant, \( \theta \) is the constant (absolute) temperature and \( m^g \) is the molecular mass of the gas. From the basic definition of the compaction of the gas in (3.28) we obtain

\[
\rho^f = \rho^0 e^{-\epsilon^f_v} \Rightarrow p = p_0 e^{-\epsilon^f_v} \quad \text{with} \quad p_0 = \frac{R \theta^f \rho^0}{m^g}
\]

where \( p_0 \) is the initial gas-pressure (normally considered as the atmospheric pressure). We remark that the linearized response of the gas phase may be represented as

\[
p = -K^f \epsilon^f_v \quad \text{with} \quad K^f = \frac{p_0}{e^{-\epsilon^f_v}}
\]

where \( K^f \) is the compression modulus of the gas. Indeed, \( K^f \) increases when the gas is densified, e.g. in the extreme case we obtain \( K^f \to \infty \) when \( \epsilon^f_v \to -\infty \).

Hence, in view of (3.99) we find that the stored mechanical energy in the gas phase may be formulated in terms of the compaction of the gas-phase, i.e. \( \psi^f = \psi^f[\epsilon^f_v] \), so that

\[
\frac{\partial \psi^f}{\partial \epsilon^f_v} = -\frac{p}{\rho^f} = -\frac{R \theta}{m^g}
\]

### 3.6.7 A remark on the intrinsic fluid flow

We note that in view of the constitutive relations (3.113), (3.115) and (3.111) that

\[
s^f = 2\mu^f I^{\text{dev}} : l^f, \quad h^f_c = -\frac{\mu^f}{K^s} v^d, \quad p = -\rho^f \frac{\partial \psi^f}{\partial \epsilon^f_v} = \left\{ \rho^f = \frac{m^g}{R \theta^f} \right\} = \frac{R \theta^f}{m^g} \rho^f
\]

Upon inserting these relations into (3.105) we obtain corresponding to the compressible Navier-Stokes equation representing the fluid flow in terms of the intrinsic properties. This is formulated as

\[
\rho^f \frac{D^f y^f}{Dt} = 2\mu^f \nabla \cdot (I^{\text{dev}} : l^f) - \frac{R \theta^f}{m^g} \nabla \rho^f - \frac{\mu^f}{K^s} v^d + \rho^f g
\]
3.7 Balance relations for different types of porous media

In the following subsections we consider a couple of specific porous media specified as specializations from the developed framework of balance relations of porous media. In addition, the governing coupled solutions for the different porous media are formulated in weak form. The following theory questions define the development of this section:

1. Define and discuss the specialization to the classical incompressible solid-liquid porous medium. Which are the restrictions imposed? Elaborate, in particular, on the balance of mass in this context. Which are the conditions imposed on the volume fractions?

2. Formulate the entropy inequality in the context of the incompressible solid-liquid porous medium. Formulate and discuss the relevant constitutive relations in this context.

3. Set up the governing equations for the incompressible solid-liquid porous medium. Define the primary solution and establish the weak coupled equations corresponding to this solution.

4. Define and discuss the specialization to the compressible solid-gas medium. Which are the restrictions imposed in this case as compared to the previous one? Elaborate, in particular, on the balance of mass in this context. Which are the conditions imposed on the volume fractions?

5. Set up the governing equations for the compressible solid-gas medium. Define the primary solution and establish the weak coupled equations corresponding to this solution.

6. Formulate and discuss the consequences of the small solid deformation assumption from the kinematical and constitutive response point of view.

3.7.1 Classical incompressible solid-liquid porous medium

Let us consider as a first prototype the classical incompressible solid-liquid porous medium. To this end, we assume firstly (which was in fact already assumed in the development of the constitutive relations) that the medium is non-erosive corresponding to \( g_f = -g_s = 0 \).

Secondly, as alluded to in the title, we assume that the phases are incompressible corresponding to the compressibility strains

\[
\epsilon^s_v := 0, \quad \epsilon^f_v := 0
\]
for the solid and the fluid phases, respectively, cf. eqs. (3.27, 3.28). Hence, the intrinsic densities of the phases are stationary, i.e. we have \( \dot{\rho}^s = 0 \) and \( \dot{\rho}^f = 0 \) in this case. Thirdly it is assumed that the dissipative fluid stress is small enough to be neglected, i.e. we set \( s_f := 0 \).

### Balance of mass

In view of the aforementioned restrictions, we conclude in view of the balance of mass in (3.31) of the mixture material that we have for incompressible phases the relationship:

\[
\nabla \cdot \mathbf{v} = -\frac{1}{\rho^f} \nabla \cdot (\rho^f \mathbf{v}^d)
\]

In addition we, of course have the mass balance conditions in (3.23) for the individual phases as usual:

\[
\dot{M}^s = 0, \quad \dot{M}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = 0
\]

which may be elaborated on for the present case of incompressible phases as

\[
\dot{M}^s = 0 \Rightarrow n^s \rho^s J = n_0^s \rho_0^s \Rightarrow n^s = \{\rho^s = \rho_0^s\} = \frac{n_0^s}{J} \Rightarrow \dot{n}^s J + n^s \dot{J} = 0 \Rightarrow \\
\dot{n}^f = n^s \frac{\dot{J}}{J} = (1 - n^f) \nabla \cdot \mathbf{v}
\]

Hence, we may develop the balance of mass for the fluid phase in (3.126b) as

\[
\dot{M}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = J \dot{\rho}^f + J \dot{\rho}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = J \dot{n}^f \rho^f + J \nabla \cdot \mathbf{v} \dot{\rho}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = J (1 - n^f) \nabla \cdot \mathbf{v} \rho^f + J n^f \nabla \cdot \mathbf{v} \rho^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = J \nabla \cdot \mathbf{v} \rho^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = 0
\]

In view of sequence of developments in (3.128), we thus conclude that the balance of mass in (3.127) is indeed embedded in balance of mass for the fluid phase. In addition, from the developments in (3.128), we conclude that the conditions for the volume fractions based on the assumption of incompressible solid phase can be formulated as:

\[
n^s = \frac{n_0^s}{J}, \quad n^f = \frac{J - (1 - n_0^f)}{J} \Rightarrow \dot{n}^f = n^s \frac{\dot{J}}{J} = (1 - n^f) \nabla \cdot \mathbf{v}
\]
Balance of momentum

As to the balance of momentum, we simply specialize the momentum balance of the mixture material in (3.43) from the assumption of a non-erosive material to the equilibrium relation

\[ \sigma \cdot \nabla + \dot{\rho} g = \dot{\rho}^s \dot{v}^s + \dot{\rho}^f (\dot{v}^f + \dot{l}^f \cdot v^r) \quad \forall x \in B \]

where it may be noted the \( \sigma \) is the total stress which in turn is related to the effective (constitutive) stress \( \sigma \) and the fluid pressure \( p \) via the Terzaghi effective stress principle in (3.70), i.e. \( \sigma = \sigma - p I \)

Entropy inequality of mixture material

Based on the fact that our mixture material can be characterized with respect to the isothermal condition, incompressible phases and non-erosive material, we make the following restrictions from the general entropy inequality established in the sequel (3.124-3.130) as:

\[ \psi^s [A, \theta] \rightarrow \psi^s [A], \psi^f [\epsilon^f, \theta] \rightarrow 0, s^f \rightarrow 0, g^f \rightarrow 0 \]

We emphasize that this restriction could not have been done prior to the establishment of the entropy inequality. We rather make the restriction when the expression of the entropy inequality have been fully developed. The specialized entropy inequality is now reduced to two independent mechanisms, namely, the dissipation produced by the incompressible solid phase considered as a porous skeleton and the interaction between the phases. This is written as

\[ D^s \geq 0, D^i \geq 0 \]

where

\[ D^s = \sigma : l - n^s \rho^s \dot{v}^s \geq 0, D^i = -h_e^f \cdot v^f \geq 0 \]

As to the effective drag force, we restrict from the final expression in (3.101) with \( s^f \rightarrow 0 \) to the relation

\[ h_e^f = \nabla p - \rho^f g + \rho^f \frac{D^f v^f}{Dt} \]

Concerning the constitutive relations in the present context, we are now guided by the relation (3.109) to propose for hyperelastic response and isotropic linear Darcy interaction between the phase the constitutive response:

\[ JD^s = \frac{1}{2} S : \dot{\mathbf{C}} - \rho^s_0 \frac{\partial \psi^s}{\partial \mathbf{C}} : \dot{\mathbf{C}} = 0 \Rightarrow S = 2\rho^s_0 \frac{\partial \psi^s}{\partial \mathbf{C}} \]
3.7. BALANCE RELATIONS FOR DIFFERENT TYPES OF POROUS MEDIA

(3.136) \[ D^i = -h^i \cdot v^d \geq 0 \Rightarrow v^d = -k h^i \Rightarrow v^d = -k \left( \nabla p - \rho f \left( g - \frac{D^f v^f}{Dt} \right) \right) \]

where \( k \) is the isotropic solid-liquid permeability parameter, cf. eq. (3.113).

Weak form of governing equations

In order to resolve e.g. the effective stress \( S \) and the Darcy flow \( v^d \) pertinent to a structure problem, we consider the constitutive relations, the balance of mass, the balance of momentum formulated as a coupled set of relations. In the present case, these coupled relations may be established with respect to the current configuration \( B \), shown in Fig. 3.4, and we formulate the momentum balance of the mixture from (3.130), the balance of mass for the mixture in (3.128), and the intrinsic fluid Darcy flow in (3.136) as

\[
(\sigma - p^f) \cdot \nabla + \hat{\rho} g = (\hat{\rho}^s + \hat{\rho}^f) \hat{v} + \hat{\rho}^f \hat{v}^r + \hat{\rho}^f (l + l^r) \cdot v^r \\
\nabla \cdot v + \frac{1}{\rho^f} \nabla \cdot (\hat{\rho}^f v^r) = 0 \\
h^f_e = \nabla p - \rho^f g + \rho^f \frac{D^f v^f}{Dt} = \\
\nabla p - \hat{\rho}^f g + \hat{\rho}^f (\hat{v} + \hat{v}^r) + \hat{\rho}^f (l + l^r) \cdot v^r \overset{\text{Darcy}}{=} -\frac{n^f}{k} v^r
\]

We identify the primary variables for the coupled problem as the placement \( x = \varphi[X] \Rightarrow v = \varphi \) for the solid particles, the fluid pressure \( p \), and the relative velocity \( v^r \), i.e. the primary solution is identified as \( P = \{v, p, v^r\} \).

The idea is then to establish the solution \( P \in V \) in weak form where \( V \) is the function space associated with \( P = \{v, p, v^r\} \) so that

(3.138) \[ V \{w, \eta, w^r\} = \{w \in P, \eta \in S, w^r \in H\} \]

where in turn \( P \) is the function space containing the virtual displacement field \( w[x] \), \( S \) is the function space containing the virtual fluid pressure field \( \eta[x] \), and \( H \) is the function space containing the virtual relative (or seepage) velocity field \( w^r[x] \).

The coupled set of equations in (3.137) are recast into weak form following the standard steps: 1) Multiply eqs. (3.137) with the proper virtual field and 2) integrate
over the current domain as

\begin{align}
\int_B \sigma : l[w] \, dv + \int_B (\dot{\rho}^s + \dot{\rho}^f) \, w \cdot \dot{v} \, dv + \int_B \dot{\rho}^f w \cdot \dot{v}^r \, dv + \int_B \dot{\rho}^f w \cdot (l + l^r) \cdot v^r \, dv = \\
\int_{\Gamma} w \cdot \sigma \cdot n \, d\Gamma + \int_B w \cdot \dot{\rho} g \, dv
\end{align}

\begin{align}
\int_B \eta \rho^f \nabla \cdot \mathbf{v} \, dv + \int_B \eta \nabla \cdot (\rho^f v^d) \, dv = \int_B \eta \nabla \cdot \mathbf{v} \, dv - \int_B (\nabla \eta) \cdot v^d \, dv + \int_{\Gamma} \eta \rho^f v^d \cdot n \, d\Gamma = 0
\end{align}

\begin{align}
\int_B \mathbf{w}^r \cdot \nabla p \, dv - \int_B \rho^f \mathbf{w}^r \cdot g \, dv + \int_B \rho^f \mathbf{w}^r \cdot (\dot{v} + \dot{v}^r) \, dv + \\
\int_B \rho^f \mathbf{w}^r \cdot (l + l^r) \cdot v^r \, dv + \int_B \frac{\eta^f}{k} \mathbf{w}^r \cdot v^r \, dv = 0
\end{align}

where the last equalities were obtained using the divergence theorem.
3.7. BALANCE RELATIONS FOR DIFFERENT TYPES OF POROUS MEDIA

Upon introducing the notation:

\begin{equation}
\tilde{t} = \sigma \cdot n, \quad Q = \rho^f v^d \cdot n
\end{equation}

we may (via (3.70) and (3.140)) reestablish eq. (3.139) as

\begin{equation}
\int_B (\sigma - p) : \mathbf{t} \mathbf{w} dv + \int_B (\dot{\rho}^s + \dot{\rho}^f) \mathbf{w} \cdot \dot{\mathbf{v}} dv + \int_B \dot{\rho}^f \mathbf{w} \cdot (l + \mathbf{l}') \cdot \mathbf{v}^r dv = \\
\int_{\Gamma} \mathbf{w} \cdot \tilde{t} d\Gamma + \int_B \dot{\rho} \mathbf{w} \cdot \mathbf{g} dv \quad \forall \mathbf{w} \in P
\end{equation}

\begin{equation}
\int_B \eta \rho^f \nabla \cdot \mathbf{v} dv - \int_B \frac{1 - n_0^f}{J} (\nabla \eta) \cdot \mathbf{v}^r dv = - \int_{\Gamma} \eta Q d\Gamma \quad \forall \eta \in S
\end{equation}

\begin{equation}
\int_B \mathbf{w}^r \cdot \nabla p dv + \int_B \rho^f \mathbf{w}^r \cdot (\dot{\mathbf{v}} + \dot{\mathbf{v}}^r) dv + \int_B \rho^f \mathbf{w}^r \cdot (l + \mathbf{l}') \cdot \mathbf{v}^r dv + \\
\int_B \frac{1}{k} \left( \frac{1 - n_0^f}{J} \right) \mathbf{w}^r \cdot \mathbf{v}^r dv = \int_B \rho^f \mathbf{w}^r \cdot \mathbf{g} dv \quad \forall \mathbf{w}^r \in H
\end{equation}

We emphasize that these relation defines the solution \( \mathcal{P} \in V \).

3.7.2 Compressible solid-gas medium

Let us consider, as the second prototype, the compressible solid-gas porous medium. To this end, like in the previous case, we assume that the medium is non-erosive corresponding to \( g^f = -g^s = 0 \). In addition, it is assumed that the solid phase is incompressible whereas the fluid phase is compressible, i.e. in view of eqs, (3.27-3.28) we have now

\begin{equation}
\epsilon_s^v := 0, \quad \epsilon_f^v \neq 0
\end{equation}

It is still assumed that the dissipative fluid stress is small enough to be neglected, i.e. \( s^f := 0 \). In order to model the “fluid” compressibility, the fluid (or gas) phase is here considered as a compressible gas concerning the non-viscous pressure response, e.g. we consider gas-filled foam materials subjected to large rapid deformations, whereby the gas-pressure response is modeled using the ideal gas law (already specified in (3.121)) written as

\begin{equation}
\rho^f = \frac{m_g}{R \theta} p = \rho_0^f e^{-\epsilon_f^v} \Rightarrow p = p_0 e^{-\epsilon_f^v} \quad \text{with} \quad p_0 = \frac{R \theta}{m_g} \rho_0^f
\end{equation}
where the last equality was obtained using the definition of the fluid phase compressibility in (3.28). We also find that from (3.143) that

\[
\frac{\dot{\rho}_f}{\rho_f} = -\dot{\epsilon}_v = \frac{m^g \dot{p}}{R \theta} = \frac{\dot{p}}{p}
\]

### Balance of mass

In view of the aforementioned restrictions, we conclude in view of the balance of mass in (3.43) of the mixture material that we have for incompressible phases the relationship:

\[
\nabla \cdot \mathbf{v} - n^f \dot{\epsilon}_v = -\frac{1}{\rho_f} \nabla \cdot (\rho^f \mathbf{v}^d)
\]

Let us also in this case elaborate on the mass balance conditions in (3.23) for the individual phases written as

\[
\dot{M}^s = 0, \quad \dot{M}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = 0
\]

where it is noted that \(\dot{M}^s = 0\) yields (just as in the case incompressible phases) the conditions

\[
\dot{M}^s = 0 \Rightarrow n^s \rho^s J = n^s_0 \rho^s_0 \Rightarrow n^s = \{\rho^s = \rho^s_0\} = \frac{n^s_0}{J} \Rightarrow \dot{n}^s J + n^s \dot{J} = 0 \Rightarrow
\]

\[
\dot{n}^f = n^s \frac{\dot{J}}{J} = (1 - n^f) \nabla \cdot \mathbf{v}
\]

Hence, the balance of mass for the fluid phase in (3.146) may be expanded as

\[
\dot{M}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = J \dot{\rho}^f + J \dot{\rho}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) =
\]

\[
J n^s \dot{\rho}^f + J n^s \dot{\rho}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) =
\]

\[
J (1 - n^f) \nabla \cdot \mathbf{v} + J n^f \nabla \cdot \mathbf{v}^d + J n^f \dot{\rho}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) =
\]

\[
J \nabla \cdot \mathbf{v} + J n^f \dot{\rho}^f + J \nabla \cdot (\rho^f \mathbf{v}^d) = 0
\]

\[
\Rightarrow \nabla \cdot \mathbf{v} + n^f \frac{\dot{\rho}^f}{\rho^f} + \frac{1}{\rho^f} \nabla \cdot (\rho^f \mathbf{v}^d) = 0
\]

which indeed is the balance of mass specified in (3.145). In addition, from the developments in (3.147) and (3.148) we conclude the conditions for the volume fractions from the assumption of incompressible solid phase as:

\[
n^s = \frac{n^s_0}{J}, \quad n^f = \frac{J - (1 - n^f_0)}{J} \Rightarrow \dot{n}^f = n^s \frac{\dot{J}}{J} = (1 - n^f) \nabla \cdot \mathbf{v}
\]


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Balance of momentum

Concerning the balance of momentum for the mixture material in (3.43) we obtain, from the assumption of a non-erosive material and the fact that the gas density \( \hat{\rho}_f \ll \hat{\rho}_s \), the equilibrium relation for the mixture

\[
(\sigma - p\mathbf{1}) \cdot \nabla + \hat{\rho}_f \mathbf{g} = \hat{\rho}_s \hat{\dot{v}}^s + \hat{\rho}_f \left( \hat{\dot{v}}^f + \mathbf{u}^f \cdot \mathbf{v}^r \right) = \hat{\rho}_s \hat{\dot{v}}^s \quad \forall x \in \mathcal{B}
\]

where the total stress \( \sigma = \sigma - p\mathbf{1} \) was directly replaced by the effective (constitutive) stress \( \sigma \) and the fluid pressure \( p \) via the Terzaghi effective stress principle in (3.70). We note the significant simplification of the inertia forces that can be made as compared to the previous prototype!

Entropy inequality

In this case we make the following restrictions from the entropy inequality developed in the sequel (3.94-3.98):

\[
\psi^s[A, \theta] \rightarrow \psi^s[A], \quad \psi^f[\epsilon^f, \theta] \rightarrow \psi^f[\epsilon^f], \quad s^f \rightarrow 0, \quad g^f \rightarrow 0
\]

It is emphasized that these restrictions could not have been done prior to the establishment of the entropy inequality! Like in the case of incompressible phases, the specialized entropy inequality is reduced to two independent mechanisms, i.e.

\[
D^s = \sigma : \mathbf{l} - n^s \hat{\rho}_s \hat{\dot{v}}^s \geq 0, \quad D^i = -h_e^f \cdot \mathbf{v}^d \geq 0
\]

As to the effective drag force we restrict from the final expression in (3.101) with \( s^f \rightarrow 0 \) to the relation

\[
h_e^f = \nabla p - \hat{\rho}_f \mathbf{g} + \rho_f \frac{D^f \mathbf{v}^f}{Dt} \approx \nabla p
\]

where we assumed that \( \nabla p \) is the dominating term (since \( \rho_f \) is very small).

As to the constitutive relations for the solid phase response we obtain from (3.109) for hyperelastic response and isotropic linear Darcy interaction between the phases the constitutive response:

\[
JD^s = \frac{1}{2} \mathbf{S} : \dot{\mathbf{C}} - \hat{\rho}_0^s \frac{\partial \psi^s}{\partial \mathbf{C}} : \dot{\mathbf{C}}:0 \Rightarrow \mathbf{S} = 2\hat{\rho}_0^s \frac{\partial \psi^s}{\partial \mathbf{C}}
\]

\[
D^i = -h_e^f \cdot \mathbf{v}^d \geq 0 \Rightarrow \mathbf{v}^d = -k h_e^f \Rightarrow \mathbf{v}^d = -k \left( \nabla p - \rho_f \left( g - \frac{D^f \mathbf{v}^f}{Dt} \right) \right)
\]

where \( k \) is the isotropic permeability parameter of the solid-gas interaction.
Weak form of governing equations

In order to formulate the coupled solid-gas interaction problem we combine the balance of mass, the balance of momentum and the intrinsic fluid flow as a coupled set of equations. This is formulated with respect to the current configuration $\mathcal{B}$ as

$$(\sigma - p \mathbf{1}) \cdot \nabla + \hat{\rho}^s \mathbf{g} = \hat{\rho}^s \dot{\mathbf{v}}$$

$$\nabla \cdot \mathbf{v} - n^f \dot{\epsilon}_v^f + \frac{1}{\rho^f} \nabla \cdot (\rho^f \mathbf{v}^d) = \nabla \cdot \mathbf{v} + n^f \frac{\dot{p}}{p} - \frac{1}{\rho^f} \nabla \cdot (\rho^f \mathbf{v}^d) = 0$$

$$h_e^f = \nabla p = -\frac{n^f}{k} \mathbf{v}^r$$

where the constitutive relation for the gas densification (3.121) was embedded into the balance of mass. Like in the previous fully incompressible problem, the problem is formulated in the solid phase placement $\mathbf{x} = \varphi(\mathcal{X}) \Rightarrow \mathbf{v} = \dot{\varphi}$, the fluid pressure $p$, and the relative velocity $\mathbf{v}^r$. Hence, we formally identify the primary solution as $P = \{\mathbf{v}, p, \mathbf{v}^r\}$.

However, in the present case the we reduce the problem via simple elimination between the two last equations in (3.155), i.e. we may eliminate the seepage velocity $\mathbf{v}^r$ so that $P \rightarrow P = \{\mathbf{v}, p\}$ and (3.155) is rewritten as

$$(\sigma - p \mathbf{1}) \cdot \nabla + \hat{\rho}^s \mathbf{g} = \hat{\rho}^s \dot{\mathbf{v}}$$

$$\nabla \cdot \mathbf{v} - n^f \dot{\epsilon}_v^f + \frac{1}{\rho^f} \nabla \cdot (\rho^f \mathbf{v}^d) = \nabla \cdot \mathbf{v} + \frac{J - (1 - n^f_0) \frac{\dot{p}}{p}}{J} - \frac{1}{\rho^f} \nabla \cdot (k \rho^f \nabla p) = 0$$

where the expression for the fluid phase volume fraction $n^f$ in (3.149) was inserted in the last equality.

The relations (3.156) are formulated in weak form so that the solution $\mathcal{P} \in V$ where $V$ is a function space defined by

$$(3.157) \quad V[\mathbf{w}, \eta] = \{\mathbf{w} \in P, \eta \in S\}$$

where in turn $P$ is the function space containing the virtual displacement field $\mathbf{w}[\mathbf{x}]$, $S$ is the function space containing the virtual fluid pressure field $\eta[\mathbf{x}]$.

The coupled set of equations in (3.156) are recast into weak form following the steps: Multiply eqs (3.156) with the proper virtual field and integration over the current domain.
This is carried out using the divergence theorem as
\[ (3.158) \]
\[
\int B (\sigma - p 1) : l(w) dv + \int B \dot{\rho}^s w \cdot \dot{v} dv = \int \Gamma w \cdot \vec{\sigma} \cdot n d\Gamma + \int B w \cdot \dot{\rho} g dv
\]
\[
\int B \eta \rho^f \nabla \cdot \dot{v} dv + \int B \rho^f J - (1 - n_0^f) \frac{\dot{p}}{p} dv - \int B \eta \nabla \cdot (k \rho^f \nabla p) dv =
\]
\[
\int B \eta \rho^f \nabla \cdot \dot{v} dv + \int B \rho^f J - (1 - n_0^f) \frac{\dot{p}}{p} dv + \int B k \rho^f (\nabla \eta) \cdot (\nabla p) dv + \int \eta v^d \cdot nd\Gamma = 0
\]
where the last equation in \[(3.156)\] (balance of mass) was premultiplied by the density \( \rho^f \) prior to the integration in order to avoid the complicated application of the divergence theorem. Let us also introduce the standard notation for the traction and the normal gas flow on the external boundary, i.e.
\[ (3.159) \]
\[
\vec{t} = \sigma \cdot n, \quad Q = \rho^f v^d \cdot n
\]
Hence, in view if \[(3.158)\] the solution \( \mathcal{P} \in V \) is established from the coupled set of relations:
\[ (3.160) \]
\[
\int B (\sigma - p 1) : l(w) dv + \int B \dot{\rho}^s w \cdot \dot{v} dv = \int \Gamma w \cdot \vec{t} d\Gamma + \int B \dot{\rho}^s w \cdot g dv \forall w \in P
\]
\[
\int B \eta \rho^f \nabla \cdot \dot{v} dv + \int B \rho^f J - (1 - n_0^f) \frac{\dot{p}}{p} dv + \int B k \rho^f (\nabla \eta) \cdot (\nabla p) dv =
\]
\[- \int \Gamma \eta Q d\Gamma \forall \eta \in S
\]

### 3.7.3 Restriction to small solid deformations - Classical incompressible solid-liquid medium

In this subsection, we outline the formulation and consequences of the assumption of small solid deformations pertinent to the classical incompressible solid-liquid medium. We emphasize that the assumption of small deformation concerns merely the solid phase response. As to the fluid response, the assumption of small deformation is of no relevance.

In the case of small solid deformations we make the simplifications: The current domain is the same as the solid reference domain, i.e. we have that
\[ (3.161) \]
\[
B \rightarrow B_0 \Rightarrow \nabla X \rightarrow \nabla, \quad dV \rightarrow dv
\]
CHAPTER 3. A HOMOGENIZED THEORY OF POROUS MEDIA

The small deformation solid displacement \( u \) is the solid velocity

\[
(3.162) \quad v \rightarrow \dot{u} \Rightarrow l = \dot{u} \otimes \nabla
\]

whereby the solid spatial velocity gradient becomes equal to the rate of displacement gradient. In addition, the deformation gradient and its Jacobian tends to unity, i.e.

\[
(3.163) \quad F \rightarrow 1, \quad J \rightarrow 1
\]

and the engineering strain rate becomes equal to the linearized Lagrange strain tensor, i.e.

\[
(3.164) \quad \dot{\varepsilon} \rightarrow \frac{1}{2} \dot{C} = \frac{1}{2} (l + l^t) = (\dot{u} \otimes \nabla)^{\text{sym}}, \quad \dot{J} = \nabla \cdot v \rightarrow \nabla \cdot \dot{u} = 1 : \dot{\varepsilon} = \dot{\varepsilon}_v
\]

**Balance of mass**

From (3.125), corresponding to the case of incompressible phases, the mass balance reads

\[
(3.165) \quad \dot{\varepsilon}_v = -\frac{1}{\rho^f} \nabla \cdot (\rho^f v d)
\]

In view of (3.127) and (3.128), we now have the conditions for the volume fractions

\[
(3.166) \quad n^s = n^s_0, \quad n^f = n^f_0 \Rightarrow \dot{n}^f = n^s \dot{J} = (1 - n^f) \dot{\varepsilon}_v
\]

**Constitutive equations**

On the basis of the solid entropy inequality (3.94), we now obtain in the small deformation case with \( \psi^s[C, \theta] \rightarrow \psi^s[\varepsilon] \) the condition with respect to hyperelasticity

\[
(3.167) \quad D^s = \sigma : \dot{\varepsilon} - \dot{\rho}_0 \frac{\partial \psi^s}{\partial \varepsilon} : \dot{\varepsilon} = 0 \Rightarrow \sigma = \dot{\rho}_0 \frac{\partial \psi^s}{\partial \varepsilon}
\]

where \( \sigma[\varepsilon] \) is now the small deformation effective stress tensor. As a prototype for small deformations, we may choose the linear elastic response corresponding to

\[
(3.168) \quad \dot{\rho}_0 \psi^s = \frac{1}{2} \varepsilon : E : \varepsilon \Rightarrow \sigma = E : \varepsilon
\]

where \( E \) is the elastic (constant) stiffness modulus tensor.
Weak form of governing equations

From (3.141) we now reconsider the governing equations in the context of small deformations as

\[
(\sigma - p\mathbf{1}) \cdot \nabla + \dot{\rho} \mathbf{g} = (\dot{\rho}^s + \dot{\rho}^f) \dot{\mathbf{u}} + \dot{\rho}^f \dot{\mathbf{v}}^r + \dot{\rho}^f (\mathbf{l} + \mathbf{l}^r) \cdot \mathbf{v}^r
\]

\[
\dot{\epsilon} + \frac{1}{\rho^f} \nabla \cdot (n_0^f \rho^f \mathbf{v}^r) = 0
\]

\[
\nabla p - \rho^f \mathbf{g} + \rho^f (\dot{\mathbf{v}} + \dot{\mathbf{v}}^r) + \rho^f (\mathbf{l} + \mathbf{l}^r) \cdot \mathbf{v}^r = -\frac{n_0^f}{k} \mathbf{v}^r
\]

corresponding to the weak form with the solution \( \mathbf{P} \in V \times S \times H \) written as

\[
\int_{\mathcal{B}} (\sigma - p\mathbf{1}) \cdot \mathbf{w} \, dv + \int_{\mathcal{B}} (\dot{\rho}^s + \dot{\rho}^f) \mathbf{w} \cdot \dot{\mathbf{v}} \, dv + \int_{\mathcal{B}} \dot{\rho}^f \mathbf{w} \cdot \mathbf{v}^r \, dv + \int_{\mathcal{B}} \dot{\rho}^f (\mathbf{l} + \mathbf{l}^r) \cdot \mathbf{v}^r \, dv = 0
\]

\[
\int_{\Gamma} \mathbf{w} \cdot \tau \, d\Gamma + \int_{\mathcal{B}} \mathbf{w} \cdot \dot{\rho} \mathbf{g} \, dv \quad \forall \mathbf{w} \in V
\]

\[
\int_{\mathcal{B}} n^f_0 \rho^f (\nabla \eta) \cdot \mathbf{v}^r \, dv = -\int_{\Gamma} \eta \mathbf{Q} \, d\Gamma \quad \forall \eta \in S
\]

\[
\int_{\mathcal{B}} \mathbf{w}^r \cdot \nabla p \, dv + \int_{\mathcal{B}} \rho^f \mathbf{w}^r \cdot (\dot{\mathbf{v}} + \dot{\mathbf{v}}^r) \, dv + \int_{\mathcal{B}} \rho^f \mathbf{w}^r \cdot (\mathbf{l} + \mathbf{l}^r) \cdot \mathbf{v}^r \, dv + \int_{\mathcal{B}} n_0^f \mathbf{w}^r \cdot \mathbf{v}^r \, dv = 0
\]

\[
\int_{\mathcal{B}} \rho^f \mathbf{w}^r \cdot \mathbf{g} \, dv \quad \forall \mathbf{w}^r \in H
\]

3.7.4 Restriction to small solid deformations - Undrained quasi-static incompressible solid-liquid medium

In this subsection, we outline the formulation and consequences of the assumption of small solid deformations pertinent to the classical quasi-static incompressible solid-liquid medium at undrained conditions, corresponding to \( \mathbf{v}^d := 0 \). This assumption is of relevance for highly impermeable porous materials under “rapid” loading, where the Darcian flow has no time to develop, cf. also the adiabatic thermal flow condition in thermodynamics.

Like in the previous sub-section, in the case of small solid deformations we make the simplifications: The current domain is the same as the solid reference domain, i.e. we have that \( \mathcal{B} \to \mathcal{B}_0 \Rightarrow \nabla_X \to \nabla, dV \to dv \), and the small deformation solid displacement \( \mathbf{u} \) is the solid velocity \( \mathbf{v} \to \dot{\mathbf{u}} \Rightarrow \mathbf{l} = \dot{\mathbf{u}} \otimes \nabla \). We also have that the deformation gradient
and its Jacobian tends to unity, i.e. $F \rightarrow 1$, $J \rightarrow 1$ whereby the engineering strain rate becomes equal to the linearized Lagrange strain tensor, i.e. 

$$\dot{\varepsilon} \rightarrow \frac{1}{2} \dot{C} = \frac{1}{2} (l + l^t) = (\dot{u} \otimes \nabla)^{sym}, \quad \dot{J} = \nabla \cdot \dot{v} \rightarrow \nabla \cdot \dot{u} = 1 : \dot{\varepsilon} = \dot{\varepsilon}_v \quad (3.171)$$

In the undrained situation we have $v^d := 0$, which due to the balance of mass in $(3.165)$ for the completely incompressible case corresponds to the kinematical constraint

$$\dot{\varepsilon}_v = \nabla \cdot v = 0 \quad (3.172)$$

In view of $(3.166)$ and $(3.172)$, we now have the conditions for the volume fractions $n^s = n_0^s, n^f = n_0^f$ and $\dot{n}^f = 0$, i.e. the volume fraction are constant in time.

On the basis of the solid entropy inequality $(3.167)$, the small deformation case $\psi^s[C, \theta] \rightarrow \psi^s[\varepsilon]$ yields the constitutive equation

$$D^s = \sigma : \dot{\varepsilon} - \dot{\rho}_0^s \frac{\partial \psi^s}{\partial \varepsilon} : \dot{\varepsilon} = 0 \Rightarrow \sigma = \dot{\rho}_0^s \frac{\partial \psi^s}{\partial \varepsilon} \quad (3.173)$$

where $\sigma[\varepsilon]$ is now the small deformation effective stress tensor. As a prototype for small deformations, we choose linear elastic response corresponding to

$$\dot{\rho}_0^s \psi^s = \frac{1}{2} \varepsilon : E : \varepsilon \Rightarrow \sigma = E : \varepsilon \quad (3.174)$$

where $E$ is the elastic (constant) stiffness modulus tensor.

We may now reconsider the governing equations in the present context written as

$$(\sigma - p1) \cdot \nabla + \dot{\rho}g = 0 \quad \dot{\varepsilon}_v = 0 \quad (3.175)$$

corresponding to the weak form with the solution $P \in V \times S$ written as

$$\int_B \sigma : l[w] dv - \int_B p \nabla \cdot w dv = \int_{\Gamma} w \cdot \mathbf{T}d\Gamma + \int_B w \cdot \dot{\rho}g dv \forall w \in V$$

$$\int_B \rho^f \eta \nabla \cdot v dv = 0 \forall \eta \in S \quad (3.176)$$

whereby it becomes evident that the fluid pressure is nothing by a Lagrangian multiplier field pertinent to the constraint $\dot{\varepsilon}_v = 0$. 

3.7. BALANCE RELATIONS FOR DIFFERENT TYPES OF POROUS MEDIA

A penalized formulation of the undrained case

As an alternative to the formulation outlined in the previous sub-section, we may consider the problem of undrained fluid flow conditions reformulated in terms of an incompressible solid phase whereas the fluid phase is considered compressible in a fictitious fashion, i.e. it is assumed that \( \epsilon_s^v := 0 \) and \( \epsilon_f^v \neq 0 \) whereby the balance of mass in view of (3.145) reads

\[
\nabla \cdot \mathbf{v} - n_f \dot{\epsilon}_v^f = 0
\]

Let us next consider the gas densification in terms of a compressibility modulus \( k_f \) defined so that

\[
\dot{\epsilon}_v^f = -\frac{\hat{p}}{k_f}
\]

Clearly, a penalized formulation of (3.177) with respect to the undrained case is obtained as \( k_f \to \infty \) leading to \( \dot{\epsilon}_v^f \to 0 \) and from (3.178) one obtains \( \nabla \cdot \mathbf{v} \to 0 \), corresponding to a limiting undrained situation. Moreover, from (3.177) we also find that the fluid pressure rate \( \dot{p} \) is obtained as

\[
\dot{p} = -\frac{k_f}{n_f} \nabla \cdot \mathbf{v}
\]

Upon integrating the fluid pressure rate in time in an implicit fashion we obtain

\[
p = n_p - \Delta t \frac{k_f}{n_f} \nabla \cdot \mathbf{v}
\]

where due to the implicit integration we have \( \nabla \cdot \mathbf{v} = n^{+1}(\nabla \cdot \mathbf{v}) \). We thus conclude that the coupled \( \mathbf{u} - p \) problem in (3.176) may be reduced to a pure displacement formulation as

\[
\int_B \left( \sigma[v] - \left( n_p - \Delta t \frac{k_f}{n_f} \nabla \cdot \mathbf{v} \right) \mathbf{1} \right) : l[w] dv = \int_{\Gamma} \mathbf{w} \cdot \mathbf{t} d\Gamma + \int_B \mathbf{w} \cdot \hat{\rho} g dv \ \forall \mathbf{w} \in V
\]

The unknown velocity field \( \mathbf{v}[\mathbf{x}] \) is solved from in a time stepping procedure using the proposed implicit integration in (3.180). The pressure field is then conveniently updated using the relation (3.179).
3.8 Numerical procedures - Classical incompressible solid-liquid porous medium

In this section we outline the numerical solution strategy to the classical incompressible solid-liquid porous medium. In particular, the temporal integration of the momentum balance relations is formulated as an explicit integration step in the spirit of the central difference scheme and the forward Euler method. Likewise, the balance of mass is integrated using the backward Euler method. The discretization of the problem is proposed and the finite element out-of-balance force vectors are established. It is proposed to resolve the coupled problem using a staggered solution procedure where the fluid pressure is considered constant over the mechanical step whereas the configuration is considered fixed over the mass balance step. Finally, a 1D linear assignment problem is outlined.

3.8.1 Temporal integration

In order to integrate in time the coupled boundary value problem we shall consider the two problems separated. We thereby consider the time interval \( I \) subdivided into a finite set of \( N \) time steps so that \( I = \{ 0, n \Delta t, \ldots, N \Delta t \} \). Typically, as a result of the time integration procedure we advance the solution step-wise from one time-step \( n \Delta t \) with known values to the updated time \( t = (n+1) \Delta t \). In the subsequent representation of the integrated solution, we omit the subscript \( \cdot n+1 \) for quantities at the updated time \( t_{n+1} \). In this fashion, we may consider two different types of integration strategies: one explicit integration related to the momentum balance relation and another implicit step related to the mass balance relation.

Momentum balance - Explicit integration step

As the point of departure, let us reconsider the governing balance relations (3.137) restated with respect to the reference configuration, whereby these equations are pre-multiplied by the "\( J \)" due to the change of integration domain. Hence, the governing relations are rewritten in terms of the "mechanical problems"

\[
J \left( \frac{T}{J} - p \mathbf{1} \right) \cdot \nabla + \left( \hat{\rho}_0 + M^f \right) \mathbf{g} = \left( M^s + M^f \right) \mathbf{\dot{v}} + M^f \mathbf{\ddot{v}} + M^f (\mathbf{l} + \mathbf{r}) \cdot \mathbf{v}^r \\
J \nabla \rho - J \rho^f \mathbf{g} + J \rho^f (\mathbf{\dot{v}} + \mathbf{\ddot{v}}^r) + J \rho^f (\mathbf{l} + \mathbf{r}) \cdot \mathbf{v}^r = -\frac{J n_f}{k} \mathbf{v}^r
\]

(3.182)

and the "mass balance problem" as

\[
J \nabla \cdot \mathbf{v} + \frac{J}{\rho^f} \nabla \cdot (\hat{\rho}^f \mathbf{v}^r) = \dot{J} + \frac{J}{\rho^f} \nabla \cdot (\hat{\rho}^f \mathbf{v}^r) = 0
\]

(3.183)
Concerning the integration of the mechanical problems \((3.182ab)\) we shall consider the well known explicit central difference scheme, where the solution is advanced in an explicit step applied to the (structural) or equilibrium equation. To this end, let us consider the central difference scheme applied to the acceleration in the momentum balance relation \((3.182a)\) whereas a standard forward Euler integration step is proposed from the relative acceleration \(\dot{v}^r\) in \((3.182b)\). This yields the integrated placement \(\varphi\) and the integrated relative velocity \(v^r\) from the Taylor series expressions

\[
\varphi = n\varphi + n v \Delta t + \frac{1}{2} n \dot{v} \Delta t^2
\]

\[
v^r = n v^r + n \dot{v}^r \Delta t
\]

where the “current” solid displacement velocity \(n v\) is defined with respect to the current and the previous time steps as

\[
nv := \frac{1}{n \Delta t + \Delta t} \left( \frac{\Delta t}{n \Delta t} (n \varphi - n^{-1} \varphi) + \frac{n \Delta t}{\Delta t} (\varphi - n \varphi) \right)
\]

Moreover, in eqs. \((3.184)\) and \((3.185)\) the accelerations \(n \dot{v}\) and \(n \dot{v}^r\) are established from the “inertia”, updated at \(t = n t\) written as

\[
(M^s + M^f) n \dot{v} = \left[ J \left( \frac{T}{J} - p I \right) \cdot \nabla + (\hat{\rho}_s^f + M^f) g \right]_{t=n t} - \left[ M^f n \dot{v}^r + M^f n (l + l^r) \cdot n v^r \right]_{t=n^* t}
\]

\[
J \rho^f n \dot{v}^r = \left[ -\frac{J n_f}{k} n v^r - J \nabla p + J \rho^f g \right] - \left[ J \rho^f \left( n \dot{v} + J \rho^f n (l + l^r) \cdot n v^r \right) \right]_{t=n^* t}
\]

whereby due to the presence of the non-linear convective terms of the inertia the integrated solution is characterized in an implicit fashion. In practice, the implicit integrated solution involving the convective term \(n (l + l^r) \cdot n v^r\) is established using a simple fix-point iteration procedure, as alluded to by the specification \(t = n^* t\) in \((3.187)\). However, concerning the establishment of the placement \(\varphi\) in \((3.184)\) we note the explicit character of the integrated solution, whereby in view of \((3.185)\) it suffices to establish the acceleration at the central point for the advancement of one step. In this advancement, the velocity is nothing but the mean velocity over the last and the current time-steps (where \(n \Delta t\) is time step-size of the last time-step). We also note that the displacement velocity \(n v\) implicitly involves the current placement.
Mass balance - Implicit integration step

In order to integrate the mass balance relation, we apply backward Euler differences to the volumetric deformation rate $\dot{J}$ of the mixture. This is formulated as

$$\dot{J} \simeq \frac{\Delta J}{\Delta t} = \frac{J - nJ}{\Delta t}$$

whereby the integrated mass balance relation in (3.183) becomes

$$\Delta J + \Delta t \frac{J}{\rho_f} \nabla \cdot (\rho_f^e v^e) = 0$$

where due to the fully implicit step all the involved variables are established at the updated time, e.g. $J = n+1 J$.

### 3.8.2 Finite element approximations

**Placement and displacement fields**

Let us next introduce finite element approximations to the solid placement field as $\varphi[X]$, where a finite element subdivision of the region $B_0$ into elements $B_{0e}$, $e=1,...,\text{NEL}$ is made. It is assumed that each element has the interpolation

$$\varphi = \sum_{I=1}^{\text{NODE}} N^I[X] \varphi^I \Rightarrow w = \delta \varphi = \sum_{I=1}^{\text{NODE}} N^I[X] w^I \Rightarrow$$

$$l[w] = \sum_{I=1}^{\text{NODE}} w^I \otimes g^I \text{ with } g^I = G^I \cdot F^{-1} \text{ and } G^I = \frac{\partial N^I}{\partial X}$$

where $N^I[X]$ are the element interpolation functions and $\varphi^I$ are the corresponding element nodal placements.

**Fluid pressure field**

The fluid-pressure field is approximated in the same way, i.e. it is assumed that each element has the interpolation

$$p = \sum_{I=1}^{\text{NODE}} N^I[X] p^I \Rightarrow \eta = \sum_{I=1}^{\text{NODE}} N^I[X] \eta^I \Rightarrow \nabla p \sum_{I=1}^{\text{NODE}} p^I g^I, \nabla \eta = \sum_{I=1}^{\text{NODE}} \eta^I g^I$$

where $p^I$ are the element nodal fluid pressures.
3.8. NUMERICAL PROCEDURES - CLASSICAL INCOMPRESSIBLE SOLID-LIQUID POROUS MEDIUM

Seepage field

In the present text, we shall consider the seepage field chosen as piecewise constant (within each \( B_{0e} \)) such that

\[
\nu^r = \sum_{e=1}^{\text{NEL}} \chi_e[X] \nu^r_e \Rightarrow w^r = \sum_{e=1}^{\text{NEL}} \chi_e[X] w^r_e
\]

where \( \nu^r_e \) is the element seepage (or relative) velocity and \( \chi_e \) is defined as

\[
\chi_e[X] = \begin{cases} 
1 & \text{iff } X \in B_{0e} \\
0 & \text{otherwise}
\end{cases}
\]

3.8.3 Finite element equations

Upon inserting the discretization into the weak forms of (3.141) for the momentum and mass balances with due consideration to the temporal integration, we obtain the discretized formulation for the accelerations \( \ddot{\nu} \) and \( \dot{\nu}^r \) in (3.187) at the current step \( n_t \) as

\[
M^n \ddot{\nu} + b^{\text{mec}}{\nu} \cdot \dot{\nu}^r + M^{\text{conv}} \nu^r \cdot \dot{\nu}^r \cdot \nu = 0 \quad \text{with} \quad b^{\text{mec}} = \sum_{e=1}^{\text{NEL}} \left( b^{\text{mec}}_e - f^e_{\text{ext}} \right)
\]

\[
M_e \ddot{\nu}^r + c_e = 0 \quad e = 1, 2, 3, ..., \text{NEL}
\]

where \( M \) is the (solid) mass matrix and \( \ddot{\nu} \) is the displacement vector of the complete finite element structure. We also establish of the integrated mass balance relation (using implicit integration in eq. (3.189)) at the updated time \( n+1t \) as

\[
b^{\text{mass}}[\varphi, p] = \sum_{e=1}^{\text{NEL}} b^{\text{mass}}_e = 0
\]

The involved contributions in the finite element equations (3.194), (3.195) and (3.203) are defined with respect to the mechanical problems (3.194, 3.195) in terms of the nodal force vectors

\[
b^{\text{mec}}_e = \sum_{I=1}^{\text{NODE}} \int_{B_{0e}} \left( \tau - Jp^I \right) \cdot g^I dV
\]

\[
M^n \ddot{\nu} = \sum_{e=1}^{\text{NEL}} \left( \int_{B_{0e}} \left( M^s + M^f \right) N^I N^J \dot{\nu}^I dV \right)
\]
\begin{equation}
M^{\text{conv}} = \sum_{e=1}^{\text{NEL}} \left( \sum_{I=1}^{\text{NODE}} A^I_0 \int_{B_{0e}} M^I N^I (\dot{\textbf{e}}^r_e + (\textbf{l} + \textbf{U}) \cdot \textbf{v}^r_e) dV \right) \tag{3.199}
\end{equation}

\begin{equation}
f^{\text{ext}}_e = \sum_{I=1}^{\text{NODE}} A^I_0 \left[ \int_{\Gamma_{0e}} N^I \textbf{t} \cdot d\Gamma_0 + \int_{B_{0e}} N^I (M^s + M^f) \textbf{g} dV \right] \tag{3.200}
\end{equation}

\begin{equation}
M_e \dot{\textbf{v}}_e = \int_{B_{0e}} \rho^f J dV \dot{\textbf{v}}^r_e \tag{3.201}
\end{equation}

\begin{equation}
\textbf{c}_e = \int_{B_{0e}} \left( \nabla \textbf{p} - \rho^f \textbf{g} + \rho^f \dot{\textbf{v}} + \rho^f \left( \textbf{l} + \textbf{U} + \frac{n^f [J] 1}{\rho^f k^f} \right) \cdot \textbf{v}^r_e \right) J dV \tag{3.202}
\end{equation}

and with respect to the balance of mass \(3.203\) of the mixture material as

\begin{equation}
b^{\text{mass}}_e = \sum_{I=1}^{\text{NODE}} A^I_0 \left[ \int_{B_{0e}} (-\Delta t J \rho^f n^f [J] \textbf{g} \cdot \textbf{v}^r_e + \rho^f N^I \Delta J) dV + \int_{\Gamma_{0e}} \Delta t N^I Q d\Gamma \right] \tag{3.203}
\end{equation}

Moreover, conjugated with the nodal vector \(b^{\text{mass}}\), we have the structural vector of nodal fluid pressures \(p\). As to the structure of \(b^{\text{mass}}\), we note that \(b^{\text{mass}}\) is the vector of out-of-balance mass transport.

### 3.8.4 Uncoupled staggered solution strategy

It is noted that the integrated relations \(3.194\), \(3.195\) and \(3.203\) along with the Taylor series advancement \(3.184\), \(3.185\) constitute a coupled set of non-linear finite element equations, which requires generally requires the proper monolithic coupled solution strategy. However, following the work of e.g. Simo and Armero [] applied to thermal problems and Landervik and Larsson [] applied to compressible solid-gas problems, the idea of the present text is to solve the FE-discretized momentum and mass balance relations \(3.194\), \(3.195\) and \(3.203\) in an uncoupled staggered fashion. To this end, we shall consider the case of constant pressure over the mechanical step (corresponding to isothermal split in the thermal case) whereas the configuration is considered fixed over the mass balance step.

#### Constant pressure

The most straightforward formulation of the decoupling is to consider the integration of the momentum balance relation with the fluid pressure considered fixed during the time-step. This means that, in the central difference scheme outlined above, we consider in
the evaluation of the acceleration \( n\ddot{\mathbf{v}} \) with prescribed fluid pressure \( \bar{p} = n\rho \) obtained from the previous time step, whereby, pertinent to the central difference scheme, the nodal acceleration \( n\ddot{\mathbf{v}} \) is established as

\[
\ddot{\mathbf{v}} = -M^{-1} \left( b_{\text{mec}} \left( n\varphi, \bar{p} \right) + M^{\text{conv}} \left( n\dot{\mathbf{v}}^r, \mathbf{v}^r, n\ddot{\mathbf{v}} \right) \right) \Rightarrow \\
\varphi = n\varphi + \frac{n\ddot{\varphi} - \ddot{\varphi}}{\Delta t_n} \Delta t + \frac{1}{2} \left( 1 + \frac{\Delta t_n}{\Delta t} \right) n\ddot{\varphi} \Delta t^2
\]

(3.204) \( \quad \quad \quad \Rightarrow \quad \quad \quad \Rightarrow \\
\dot{\mathbf{v}}^r = -\frac{1}{M_e} c_e \left[ n\varphi, \bar{p}, n\ddot{\mathbf{v}}^r, n\dot{\mathbf{v}}^r \right] \Rightarrow \mathbf{v}^r = n\dot{\mathbf{v}}^r \Delta t_n, e = 1, 2, 3..., \text{NEL}

(3.205)

where \( M \) is the (solid) mass matrix and \( \dot{\varphi} \) is the placement nodal vector of the complete finite element structure. We also emphasize the implicit character of the integrated solution \( \{ n\dot{\mathbf{v}}, n\ddot{\mathbf{v}} \} \) due to the presence of convective terms in the representation of inertia. It may be remarked that this procedure is in fact consistent with the central difference scheme, where the solution is advanced explicitly at the central point \( n\dot{\bullet} \) with known solution quantities at \( n\dot{\bullet} \) and \( n^{-1}\dot{\bullet} \) values, cf. the relations (3.184) and (3.186).

Secondly, the mass balance problem is solved with respect to the fixed placement \( \varphi = \text{const.} \) obtained from the previous mechanical step (3.204)-(3.204), i.e.

\[
b^{\text{mass}} \left[ \mathbf{p} \right]_{\varphi = \text{const.}} = 0
\]

(3.206)

It should be emphasized this relation is generally non-linear in the fluid pressure and the handling of the problem is described below in sub-section 3.8.4. Note that the coupled problem becomes reduced to first solving a conventional mechanical step, which generates input to mass balance step in form of the mechanical deformation gradient. In turn, the mass balance step provides the fluid pressure for the mechanical step.

Handling of non-linear mass balance step

As pointed out in sub-section 3.8.4 the finite element discretized mass balance relation is considered as a separate relation with fixed configuration. The goal is to establish the fluid pressure for the next time-step in the staggered solution procedure by solving for the fluid pressure in the non-linear relation \( g[\mathbf{p}] \) defined as

\[
g[\mathbf{p}] = b^{\text{mass}}[\mathbf{p}]_{\varphi = \text{const.}} = 0
\]

(3.207)

In this relation the nodal force vector \( b^{\text{mass}} \) due to the mass balance relation is now expressed as

\[
b_{e}^{\text{mass}} = \text{NODE}_{i=1}^{A} \int_{B_{0e}} \left( \Delta t J \rho^{f} n^{f} [J] g^{f} \cdot A^{-1} \cdot (\nabla p - \rho^{f} g + \rho^{f} (n\dot{\mathbf{v}} + n\dot{\mathbf{v}}^{r})) + \rho^{f} N^{f} \Delta J \right) dV + \int_{V_{0e}} \Delta t N^{f} Q d\Gamma
\]
In this establishment it was used that the gradient of the pressure is expressed via the effective drag force and the Darcy law as expressed in (3.137), i.e.

\[
\mathbf{v}_e = -A^{-1} \cdot (\nabla p - \rho f \mathbf{g} + \rho f (\mathbf{n} \mathbf{v} + \mathbf{n} \mathbf{v}_e)) \quad \text{with} \quad A = \rho f \left( n \mathbf{l} + n \mathbf{l}^r + \frac{n [J] \mathbf{l}}{\rho f k} \right)
\]

where the accelerations \( \mathbf{n} \mathbf{v}, \mathbf{n} \mathbf{v}_e \) and the spatial velocity gradients \( \mathbf{n} \mathbf{l}, \mathbf{n} \mathbf{l}^r \) are considered known from the momentum balance step. We also note that \( A^{-1} \) may be interpreted as the dynamic seepage permeability tensor.

To solve for the pressure \( p \), a standard Newton procedure is used, where, in particular, we consider the linearization of \( g[p] \) represented as

\[
dg[p] = K dp, \quad K = \sum_{e=1}^{NEL} A K_e
\]

where the structural Jacobian matrix \( K \) is considered as an assembly of element stiffness matrices \( K_e \), which in view of (3.203) are given by

\[
K_e = \sum_{i=1}^{NODENODE} A \sum_{j=1}^{A} \left[ \int_{B_{0e}} \Delta t Jk n[ ]^f [J] g I \cdot g^T dV \right]
\]

The condition \( g = 0 \) is then resolved as defined by the local Newton iteration in the fluid pressure \( i p = i^{-1} p + i \xi \) written as

\[
\left\{ i^{-1} g + dg = i^{-1} g + i^{-1} K \xi = 0 \right\}_{i=1,NITE}
\]

where NITE is the number of iterations that are required in order to satisfy the tolerance \( |g| < \epsilon \).
Chapter 4

Theory questions and assignment problems of continuum mechanics of porous media

4.1 The theory questions

The following theory questions are involved in the course:

Section 2.1

1. Define and discuss the concept of volume fractions in relation to the micro-constituents of a two-phase mixture of solid and fluid phases related to an RVE of the body.

2. Define the effective mass in terms of intrinsic and bulk densities of the phases from equivalence of mass. Discuss the issue of (in)compressibility of the basis of this discussion.

3. Define the effective (representative) velocities of the solid and fluid phases related to their micro-mechanical variations across an RVE. Discuss also the issue of “direct averaging”.

4. Based on the quite general result of stress homogenization of a one-phase material, generalize the result to the two-phase situation. Discuss the partial stresses and their relation to intrinsic stresses and micro-stress fields.

Section 3.1
1. Define and discuss the formulation of the kinematics of a two-phase mixture. Introduce the different types of material derivatives and formulate and discuss the velocity fields of the mixture.

2. Prove the formula: \[ \dot{\mathbf{j}} = J \nabla \cdot \mathbf{v} \.

Section 3.2

1. Formulate the idea of mass conservation pertinent to a one-phase mixture. Generalize this idea of mass conservation to the two-phase mixture. Discuss the main results in terms of the relative velocity between the phases.

2. Describe in words (and some formulas if necessary) the difference between the material derivative \( \dot{\rho} = D\rho(x,t)/Dt \) and the partial derivative \( \partial\rho(x,t)/\partial t \), where \( \rho \) is the density of the material.

3. Formulate the mass balance in terms of internal mass supply. Discuss a typical erosion process.

4. Formulate the mass balance in terms of the compressibility strains. Set of the total form of mass balance in terms of the saturation constraint. In view of this relationship, introduce the issue of incompressibly/compressibility of the phases in terms of the compressibility strains.

Section 3.3

1. Formulate the principle of momentum balance in total format. Formulate also the contribution of momentum from the different phases. Consider, in particular, the conservation of mass in the formulation.

2. Express the principle of momentum balance with respect to the individual phases. Focus on the solid phase reference configuration via the introduction of the relative velocity. In this, development express the localized equations of equilibrium for the different phases. Formulate your interpretation of the local interaction forces.

Section 3.4

1. Establish the principle of energy conservation of the mixture material. Discuss the involved “elements” and formulate the internal energy and kinetic energy with respect to the solid reference configuration using the relative velocity.

2. “Work out” the mechanical work rate of the mixture solid.
3. Derive the energy equation using the assumption about an ideal viscous fluid. In this context, define effective stress of Terzaghi. Discuss the interpretation of the Terzaghi stress.

Section 3.5

1. Establish the total entropy and the second law of thermodynamics in the context of a two-phase mixture material. On the basis of the involved material derivatives and principle of mass conservation, express the entropy inequality in terms of the solid reference configuration.

2. Define and discuss the suitable Legendre transformation in internal energy, Helmholtz free energy, temperature and entropy. Represent and localize the entropy inequality of the mixture material in terms of these quantities.

3. On the basis of the localized entropy for the mixture material, discuss the different components of the dissipation inequality and relate them to dissipative mechanics of the mixture material.

Section 3.6

1. Derive the state equation for the effective stress in the context of an hyper-elastic material. Formulate also the transformation between our different stress measures. Carry out the derivation with the restriction to a non-erosive mixture material. What would happen if the material would have been erosive?

2. Motivated by the dissipation $D_i$ in eq. (3.92), formulate Darcy’s law as a constitutive representation of the drag interaction between the phases. Discuss under which circumstances we can write: $\mathbf{v}^d = -k \left( \nabla p - \rho_f \left( \mathbf{g} - \frac{D_f v_f}{Dt} \right) \right)$ where $k$ is the (isotropic) permeability constant.

3. Formulate the representation of the fluid phase as a gas phase. In this development, derive the fluid compressibility!

Section 3.7

1. Define and discuss the specialization to the classical incompressible solid-liquid porous medium. Which are the restrictions imposed? Elaborate, in particular, on the balance of mass in this context. Which are the conditions imposed on the volume fractions?
2. Formulate the entropy inequality in the context of the incompressible solid-liquid porous medium. Formulate and discuss the relevant constitutive relations in this context.

3. Set up the governing equations for the incompressible solid-liquid porous medium. Define the primary solution and establish the weak coupled equations corresponding to this solution.

4. Define and discuss the specialization to the compressible solid-gas medium. Which are the restrictions imposed in this case as compared to the previous one? Elaborate, in particular, on the balance of mass in this context. Which are the conditions imposed on the volume fractions?

5. Set up the governing equations for the compressible solid-gas medium. Define the primary solution and establish the weak coupled equations corresponding to this solution.

6. Formulate and discuss the consequences of the small solid deformation assumption from the kinematical and constitutive response point of view.
4.2 Assignment - Undrained analysis of 2D specimen

Consider the solid-fluid undrained 2D specimen in Fig. 4.1. The specimen is loaded with a prescribed velocity \( \dot{r} = 2 \) in the vertical direction acting at the top of the specimen. The effective stress response is assumed to be isotropic elastic, which is represented by the stiffness modulus tensor

\[
E = 2G I^\text{sym} + K1 \otimes 1
\]

where \( G \) is the elastic shear modulus and \( K \) is the elastic bulk modulus. These are further defined in terms of Young’s modulus \( E \) and Poisson’s ratio \( \nu \) written as

\[
G = \frac{E}{2(1 + \nu)}, K = \frac{E}{3(1 - 2\nu)}
\]

The actual values of the material and geometry parameters are given in Fig. 4.1.

Two types of formulations of the undrained problem are to be compared in the assignment. The first one considers the truly incompressible case in terms of a coupled displacement-pressure problem as described in the weak form (3.176). The second formulation considers a regularized version of the truly undrained case in terms of a compressible fluid phase, represented by the fluid compressibility parameter \( k_f \to \infty \), cf. the discussion around eq. (3.181).

The following subtasks of the assignment are to be considered:

- Formulate the governing equations (including the weak form) pertinent to the undrained problem, cf. the discussion in sub-section 3.7.4. The proper formulation of the finite element discretization and establishment of coupled FE-equations must be made. It suffices to restrict to the well known Taylor-Hood element, with quadratic displacements and linear pressure variation with the element.

- Implement the FE-problem in your favorite FE-environment, e.g. Matlab, and results are to be provided so that the different formulations of the problem can be compared to each other in terms of the fluid pressure response. Note that in the limit as \( k_f \to \infty \), the two formulations should coincide!

- Analyse the effective stress, total stress and fluid pressure distribution in the undrained specimen due to the considered prescribed displacement loading. Please note the prescribed displacements at the bottom of the specimen. Discuss why the fluid pressure cannot be prescribed in the present context!

Preliminary results from the analyses are shown in Figs. 4.2, 4.3 and 4.4, corresponding to the prescribed displacement \( r = 2a/10 \). The results show that the von Mises
Figure 4.1: Considered undrained 2D–specimen with prescribed displacement loading at the top and prescribed displacements at the bottom. Set $E = 210 \text{MPa}$ and $\nu = 0.15$. Concerning the geometry, set $a = 50\text{mm}$.

stress distribution is more or less the same for the different formulations, cf. Fig. 4.2. Concerning the fluid pressure there is a difference, in particular, due to post-processing of the pressure distribution. In the incompressible formulation the fluid pressure is obtained from the element nodes from the solution vector, whereas in the compressible formulation the fluid pressure is obtained at the Gaussian integration point level of the finite elements.
Figure 4.2: von Mises stress distribution. a) undrained analysis using incompressible FE-formulation, and b) undrained analysis using compressible FE-formulation

Figure 4.3: Fluid pressure distribution. a) undrained analysis using incompressible FE-formulation, and b) undrained analysis using compressible FE-formulation

4.3 Assignment - Wave propagation in a fluid saturated pile

4.3.1 Problem definitions

Consider the linear 1D pile-problem in Fig. 4.5 subjected to the prescribed velocity \( v_0 = \dot{u}_0 = \ddot{v} \) (with the initial displacement \( u_0 = \ddot{u} = 0 \)) at the right end, whereas the
left end is considered fixed from the displacement point of view. (The restriction to 1D is given in sub-section 4.3.2 below.) The fluid pore (excess) pressure is assumed to be zero at the left and the right end of the pile. The constitutive small strain effective stress response of the pile material is (for simplicity) assumed to be linear elastic corresponding Hooke’s law. A linear Darcy law for the interaction between the phases may also be assumed. This is formulated as

\[ \sigma = E \epsilon, \quad h^f = -\frac{n f}{k} v^r \]

Evaluate the distribution of the effective stress \( \sigma[x, t] \) and the excess fluid pressure \( p[x, t] \) of the pile due to the considered dynamic loading consisting of a pulse load \( P[t] \), with the "hold" time \( t_h \). Utilize the proposed explicit temporal integration procedure for the integration of the structure finite element problem. Small strains may be assumed. As to the conservation of mass, the discuss staggered solution procedure involving a mechanical and mass balance step should be exploited.

In the development of the solution of the problem the 1D pile bar is considered discretized into a number of 1D finite elements where the proper solution variables is considered discretized. As a part of the assignment, propose the choice of regularity of the spatial discretization for the displacement \( u[x] \), the fluid pressure \( p[x] \) and the seepage velocity \( v^r[x] \).
4.3. ASSIGNMENT - WAVE PROPAGATION IN A FLUID SATURATED PILE

Figure 4.5: A 1D wave propagation problem in a fluid saturated pile.

4.3.2 Restriction to 1D

Consider the restriction to the 1D stress state of pile problem given in Fig. 4.5. The integration (computational) domain $\mathcal{B}$ is then reduced to a 1D interval so that $\mathcal{B} \rightarrow [0, L]$ leading to $x \rightarrow x$, $dv \rightarrow Adx$, where $A$ is the cross sectional area of the pile. The fact that the stress - strain state is 1-dimensional is represented as

$$\sigma \rightarrow \sigma, \quad l \rightarrow \dot{\varepsilon}$$

where $\sigma$ is the uni-axial stress and $\dot{\varepsilon}$ is the uniaxial strain rate. Likewise the traction $\bar{t} \rightarrow \bar{\sigma}[L]$.

As to kinematics we also have that:

$$u[x] \rightarrow u[x], \quad v[x] \rightarrow v[x], \quad w[x] \rightarrow w[x],$$  

$$w^r[x] \rightarrow w^r[x], \quad \bar{t} \rightarrow \bar{\sigma}[L], \quad \nabla \cdot v \rightarrow \dot{\varepsilon}, \quad \nu^r \rightarrow \nu^r, \quad \nabla p \rightarrow \frac{dp}{dx}$$

Finally, we may state the weak forms of the governing equations in the context of our 1D pile-bar problem as

$$A \int_{L} \epsilon'(\sigma - p)dx + A \int_{L} w \left( (\dot{\sigma}^s + \dot{\sigma}^f) \dot{v} + \dot{\rho}^f \dot{v}^r + \dot{\rho}^f (l + l^r) v^r \right) dx = A \left( w[L] \bar{\sigma}[L] - w[0] \bar{\sigma}[0] \right)$$

$$A \int_{L} \rho^f p' \dot{\varepsilon} dv - A \int_{L} \rho^f n_0' \frac{dp}{dx} v^r dx = -A \left( p'[L]Q[L] - p'[0]Q[0] \right)$$

$$A \int_{L} k w^r \frac{dp}{dx} dx + A \int_{L} k \rho^f w^r (\dot{\nu} + \dot{v}^r) + (l + l^r) v^r) dx + A \int_{L} n_0' w^r v^r dx = 0$$
Consider the following data for the assignment:

\( A = 0.2 \cdot 0.2m^2 \), \( L = 10m \), \( E = 21GPa \), \( n^s = 0.6 \),

\[ (4.7) \]
\[ \rho^f = 1000\frac{kg}{m^3}, \rho^s = 2600\frac{kg}{m^3}, \ k = 1 \cdot 10^{-7} \]
Bibliography


