Static correction in model order reduction techniques for multiphysical problems

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ABSTRACT

In this paper multiphysical model order reduction methods for thermomechanical problems are investigated. Different variants of modal truncation methods are outlined. A basis built of state space modes of the system represents the behavior of the system but the method requires complicated complex modes. This is why we also introduce bases built from a composition of bases of the separate uncoupled physical fields because these are simpler to build. One can improve on these bases again by inclusion of a correction for the coupling effect, for example based on a derivation from a first order perturbation analysis. These bases shows a good representation on the dynamics behavior, but generally do not give correct static results. However we want at least the static solution to be correct in order to guarantee that if the problem is quasi-static or if it converges to a steady-state static solution, one gets the exact solution. Hence one wants to add a static correction to the solution (a posteriori). That static correction can also be used to enrich the basis. In this contribution an investigation on the static residuals for different bases is performed.

Nomenclature

Matrices:

\[ M \] 2nd order time derivative system matrix
\[ C \] 1st order time derivative system matrix
\[ K \] 0th order time derivative system matrix
\[ S \] Load applied to a system
\[ E \] 1st order time derivative state space matrix
\[ A \] 0th order time derivative state space matrix
\[ L \] Output observation matrix
\[ B \] Input control matrix
\[ F \] Applied mechanical force
\[ Q \] Applied heat flux
\[ G \] Residual flexibility
\[ \Psi \] Basis built on modes of the second order problem
\[ \Phi \] Basis built on modes of first order system
\[ \Delta \Psi \] Correction of modal basis
\[ \Lambda \] Diagonal eigenvector matrix
\[ I \] Identity matrix
Vectors:
- \( u \) Mechanical DOF: deformation
- \( \theta \) Thermal DOF: temperature difference
- \( i \) Input signal
- \( r \) Residual force
- \( y \) Output signal
- \( q \) Degree of freedom
- \( x \) State space variables
- \( z \) Generalized degree of freedom
- \( \psi \) Eigenmode of a second order problem
- \( \phi \) Eigenmode of a first order problem
- \( \eta \) Modal amplitude

Scalar values:
- \( n \) Dimensions of full order system
- \( k \) Dimensions of reduced order system
- \( \omega \) Frequency
- \( T_0 \) Temperature at working point
- \( \lambda \) Eigenvalue
- \( \alpha \) Modal participation factor
- \( \zeta \) Modal damping value

Mathematical notation:
- \( i \) Imaginary number
- \( O \) Order of the error
- \( \langle \ldots \rangle_{nc} \) The uncoupled part of scalar / vector / matrix
- \( \langle \ldots \rangle^c \) The coupled part of scalar / vector / matrix
- \( \langle \ldots \rangle^t \) Transposed of vector / matrix

1 INTRODUCTION

Multiphysical models are intensively used during the design of high-tech systems and in particular microsystems. However both the complexity and large dimension of the models makes intensive use costly and clarifies the demand for reduced order models that preserve the dominant behavior of the system. This demand especially holds for models describing multiphysical behavior, because these can be very complicated whilst modeling the coupling effects is of key importance for the observed behavior. From several fields of engineering model order reduction techniques are available, but their performance is not straightforward in a multiphysical context.

Standard techniques applied to coupled problems are numerically inefficient and therefore in [1] a procedure is proposed to obtain a reduced order model starting from reduction bases that for the uncoupled physics. A reduced order model for the coupled problem is obtained by improving the ability of the uncoupled bases to represent coupling phenomena, which is done by performing a correction to the uncoupled bases that accounts for the coupling with other physics.

A specific property of a reduction technique is the ability to predict correctly the static behavior since the steady-state of systems is of prime importance in many engineering problems. In this contribution we will discuss how the correct static solution can be guaranteed in reduced order coupled problems. It gives rise to either a correction to the solution obtained from the reduced models or an augmentation of the reduction basis. This approach is illustrated on a two-way coupled thermomechanical problem, where the static coupling contains only a one-sided coupling however. This brings the possibility to perform the correction sequentially when starting from fully uncoupled bases for a separate mechanical and thermal field.

1.1 A thermomechanical application

In microsystems multiphysical behavior often becomes of interest. On the one hand several systems use multiphysical effects as the key working principle. Think for example of a commonly used actuation principle in a thermomechanical actuator. On the other hand multiphysics can have parasitic side-effects, for example when we think of thermomechanical damping in resonators. The thermomechanical behavior can be described by the following governing equation:

\[
\begin{bmatrix}
M_{uu} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\ddot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
C_{uu} & C_{u\theta} \\
C_{u\theta} & C_{\theta\theta}
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
K_{uu} & K_{u\theta} \\
K_{u\theta} & K_{\theta\theta}
\end{bmatrix}
\begin{bmatrix}
u \\
\theta
\end{bmatrix}
= \begin{bmatrix}
F \\
Q
\end{bmatrix} \tag{1}
\]

It expresses the thermomechanical equations of motion written in terms of DOF deformation \( u \) and temperature change \( \theta \). In this equation we can recognize \( M_{uu}, C_{uu}, K_{uu} \) \( F \) as mechanical mass, damping and stiffness and external force. \( C_{u\theta}, K_{u\theta} \) \( Q \) are heat capacity, thermal conductance and heat flux as can be recognized from the heat equation. The terms that are described until now can be seen to be a description for the uncoupled mechanics or thermal problem. Two terms remain that are recognized to couple the equations. Firstly we recognize the thermal expansion \( K_{u\theta} \) by which a temperature difference can influence the deformation. Secondly we recognize
$C_{\theta u}$ which describes how the strain rate, the time derivative of deformation, induces heat. From the description of the equations it is seen that a second order problem (the mechanics) is coupled to a first order problem (the thermal field). Furthermore the problem becomes especially interesting for microsystems, because when scaling the dimensions of a problem to the range of microns, the critical time constants of both physical fields can fall in the same range and thus dynamics that originates from each field can be observed. As an illustration we can look at the thermal actuator depicted in figure 1. On the lefthand side we see how two thermal actuators can be configured in a setup. On the righthand side a screen capture is given from finite element software that predicts dynamic behavior such as described in the next section.

1.2 Behavior of the full system

In this section the behavior of the thermomechanical actuator is explained. The working principle of these actuators is that an applied voltage results in resistive heating and generates a distributed heat source. Due to the thermal expansion the applied heat now results in a mechanical actuation. The system can be seen to consist of two beams with different cross-section. The thinner beam has more electric resistance and thus extends more as a result of thermal expansion. The difference in expansion leads to a net bending effect, which is seen in figure 2. The thermomechanical equations of motion can be written in general second order form as:

$$M \ddot{q} + C \dot{q} + Kq = S$$ (2)

This describes the entire system and we can investigate a certain response $y$ by measuring the DOF $q$ with $L'$. This is the response of the system to a load $S$ that is excited with $i$ and is applied according to $B$. The possibility to investigate the response of a certain output $y$ of the system with respect to an excitation $i$ applied to the system is now described by the following set of equations:

$$\begin{cases} 
S = Bi \\
M \ddot{q} + C \dot{q} + Kq = S \\
y = L'q 
\end{cases}$$ (3)

Now we will look at the transient response of this system of the deformation at the tip when a heat step loading is applied. We will find the results of figure 3. Different snap shots of the deformation are depicted. Throughout this
paper we will have in mind that a dynamic model is of importance for instance to model the transient behavior, but for the design of the actuator at the end we are interested in the possible actuation, which is the static deformation. Therefore a correct representation of the static behavior remains of interest.

1.3 First order formulation

A descriptor formulation is often used, for example to easily calculate the solution to equation 3. This formulation can be obtained by writing equation 3 in first order form. In this formulation variables are introduced that write the state of the system at an any time, which are therefore called state variables \(x\). This also induces that in general the control matrix \(B\) and observation matrix \(L\) need to be extended, leading to \(B_{ss}\) and \(L_{ss}\) respectively, where subscript \(ss\) stands for state space or first order form. The dynamic behavior of the system can now be written in terms of these state variables and their time derivatives. The general form now looks like:

\[
\begin{align*}
E \dot{x} &= Ax + S_{ss} \\
S_{ss} &= B_{ss} i \\
y &= L_{ss} x 
\end{align*}
\]  

(4)

For the thermomechanical problem, which was formulated by equation 1 we can also write such a first order form for \(E \dot{x} = Ax + S\). Moreover, we can write this expression into a symmetric form with the following operations. First we scale the identity relation for \(\dot{u}\) with \(-K_{uu}\). Second we use the relation \(C_{\theta u} = T_0 K_{u \theta}\) as is found from [2]. The third operation is to scale the heat equation by \(\frac{1}{T_0}\). Furthermore from here on we assume that mechanical damping is absent, meaning that \(C_{uu} = 0\), in order to simplify the analysis. But the developments found from the analysis can be generalized to the situation that mechanical damping is present. These steps result in:

\[
\begin{bmatrix}
-K_{uu} & 0 & 0 \\
0 & M_{uu} & 0 \\
0 & 0 & \frac{1}{T_0} C_{\theta \theta}
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
0 & -K_{uu} & 0 \\
-K_{uu} & 0 & -K_{u \theta} \\
0 & -K_{u \theta} & -\frac{1}{T_0} K_{\theta \theta}
\end{bmatrix}
\begin{bmatrix}
u \\
\dot{u} \\
\dot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
F \\
\frac{1}{T_0} Q
\end{bmatrix}
\]

(5)

Note that in design one can be interested in a small number of outputs (for instance tip displacement of the thermomechanical beam). But engineers are often interested in all physical quantities in order to ensure the reliability of the design and therefore to analyze not only the output of a device but also maximal stresses and temperatures. Therefore for the thermomechanical system we will use \(y = x\), or \(L^I = I\).
2 DIFFERENT VARIANTS OF MODAL TRUNCATION METHODS AS A REDUCTION TECHNIQUE

2.1 The idea of modal truncation

Modal truncation is a reduction technique generally applied in many fields of engineering, for example the field of mechanical engineering. The concept of modal truncation is as follows. Modes \( \psi_i \) give a solution to the homogenous equations of motion of the system and can be used to form a projection or transformation basis \( \Psi \) for the problem. For a general second order problem such as given by equation 3, we can use the following transformation:

\[
q = \Psi \eta
\]  

(6)

This writes the problem in modal coordinates. The orthogonality properties of the modes yield that the frequency response function of the system can be written as a modal summation. For a general unsymmetric problem we can distinguish the so-called left and right modes. In this case the right modes are used for a trial basis and the left modes for a test basis. For convenience we will use identical left and right modes (such as found for symmetric systems) throughout this paper in order to explain general ideas. The transfer matrix relating input to output in the frequency domain can then be written in a modal expansion as:

\[
H = \sum_{i=1}^{k} \frac{L^t \psi_i \psi_i^t B}{\lambda_i^2 - 2i\zeta_i\lambda_i\omega - \omega^2}
\]  

(7)

In modal superposition methods the number of variables in the model can be reduced by representing the dynamic behavior of the system with only a limited amount of modes. This means that the representation of the transfer function is truncated after \( k \) terms.

\[
H = \sum_{i=1}^{k} \frac{L^t \psi_i \psi_i^t B}{\lambda_i^2 - 2i\zeta_i\lambda_i\omega - \omega^2} + \sum_{i=k+1}^{n} \frac{L^t \psi_i \psi_i^t B}{\lambda_i^2 - 2i\zeta_i\lambda_i\omega - \omega^2}
\]  

\[
\approx \sum_{i=1}^{k} \frac{L^t \psi_i \psi_i^t B}{\lambda_i^2 - 2i\zeta_i\lambda_i\omega - \omega^2}
\]  

(8)

For the projection basis it implies that instead of the full modal basis \( \Psi \) a reduced basis is used that contains only the \( k \) modes. Throughout this paper we will indicate \( \hat{V} \) to be the truncated basis of \( V \) that contains only \( k \) terms. The truncated set of \( k \) corresponding DOF is indicated with \( \hat{\eta} \). This gives:

\[
\hat{q} = \hat{\Psi} \hat{\eta} = \begin{bmatrix} \psi_1 & \psi_2 & \ldots & \psi_k \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \end{bmatrix} \approx q
\]  

(9)

When the system is written in first order form we can perform a comparable operation. Now we use modes \( \phi_i \) that can be used to form a transformation basis \( \Phi \) for the problem. For a general first order problem assumed to be symmetric such as given by equation 4, we can use the transformation

\[
x = \Phi z
\]  

(10)

to write the problem in modal coordinates. The orthogonality properties again allow to write the frequency response function as a modal summation and truncating this summation after \( k \) terms we find the approximate transfer function as:

\[
H = \sum_{i=1}^{k} \frac{L^t_{sa} \phi_i \phi_i^t B_{sa}}{-\lambda_i + i\omega} \approx \sum_{i=1}^{n} \frac{L^t_{sa} \phi_i \phi_i^t B_{sa}}{-\lambda_i + i\omega}
\]  

(11)

where \( \phi_i \) and \( \lambda_i \) are respectively the (generally complex) eigenmodes and eigenvalues of the first order problem. When this transfer function is approximated the corresponding projection basis \( \Phi \) contains only the \( k \) modes, as we can see in:

\[
\hat{x} = \hat{\Phi} \hat{z} = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_k \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_k \end{bmatrix} \approx x
\]  

(12)
2.2 Different variants of modal truncation for thermomechanical systems

Now that the idea of modal truncation is explained we will discuss 4 variants of modal truncation as a reduction technique for thermomechanically coupled problems, described by equations 1 or equation 5, depending if one considers the second or the first order problem.

2.2.1 Method 1: truncation of a basis of fully coupled modes

The modes are the full modes of the coupled problem, containing both thermal DOF $\theta$ and mechanical DOF $u$. Modal truncation can be applied directly to the thermomechanical system written in state space, resulting a basis of state space modes indicated as $\Phi_{ss}$, but it is numerically inefficient to obtain these coupled modes. The basis is given by:

\[\hat{x} = \hat{\Phi}_{ss} \hat{z}_{ss}\]  

(13)

2.2.2 Method 2a: truncation of a basis of fully uncoupled modes of the second order problem

Especially when coupling remains small, either the thermal DOF or the mechanical DOF in the coupled mode show a large amount of similarity with uncoupled modes (modes of the corresponding uncoupled physical fields). Generally the uncoupled modes can be calculated directly from the equations of motion written in second order form such as given by equation 3. For the thermomechanical problem given by equation 1 a truncated basis of $k_u$ uncoupled mechanical modes $\Psi(u)$ and $k_\theta$ uncoupled thermal modes $\Psi(\theta)$ can possibly provide an efficient basis for the coupled problem in the form of:

\[\hat{q}_{nc} = \begin{bmatrix} \hat{\Psi}(u) & 0 \\ 0 & \hat{\Psi}(\theta) \end{bmatrix} \begin{bmatrix} \hat{\eta}(u) \\ \hat{\eta}(\theta) \end{bmatrix}\]  

(14)

2.2.3 Method 2b: truncation of a basis of fully uncoupled modes of the first order problem

To be consistent with the fully coupled modes that are calculated when the problem is written in first order form, we can also derive the uncoupled modes for this problem. Again when coupling remains small, either the thermal DOF or the mechanical DOF in the coupled mode show a large amount of similarity with the modes of the corresponding uncoupled physical problems written in first order form (uncoupled state space modes). In other words a truncated basis of $k_u$ uncoupled mechanical modes $\Phi(u)$ and $k_\theta$ uncoupled thermal modes $\Phi(\theta)$ could possibly provide an efficient basis for the coupled problem in the form of:

\[\hat{x}_{nc} = \begin{bmatrix} \hat{\Phi}(u) & 0 \\ 0 & \hat{\Phi}(\theta) \end{bmatrix} \begin{bmatrix} \hat{\eta}(u) \\ \hat{\eta}(\theta) \end{bmatrix}\]  

(15)

Although it seems that using this truncated basis is equivalent to using the truncating basis of the uncoupled modes in the second order, we like to especially point out that when applying the truncated basis in the first order form one also approximates the closure relation in the state-space. In other words here also the relation between velocities and time derivatives of the mechanical DOF is approximated.

2.2.4 Method 3a: truncation of a basis of corrected uncoupled modes

Although for some frequency response functions a mono-physical basis as described above indeed results in a sufficiently accurate reduced order model, especially in the representation of cross-coupling behavior it is observed that reduced models based on the truncation explained so far do not always accurately represent the dynamic behavior of the system. Therefore a correction of the uncoupled basis was proposed, see [1]. This correction was derived performing a perturbation analysis of the uncoupled basis with system matrix

\[A^c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & K_{u\theta} \\ 0 & K_{u\theta} & 0 \end{bmatrix}\]  

(16)
It results that uncoupled mechanical modes require a correction on their thermal DOF and uncoupled thermal modes require a correction on the mechanical DOF. For a single uncoupled mechanical mode $\phi_{u_i}^{nc}$ and a single uncoupled thermal mode $\phi_{\theta_i}^{nc}$, the corresponding corrections $\Delta \phi_{u_i}^{nc}$ and $\Delta \phi_{\theta_i}^{nc}$ are respectively:

$$
\Delta \phi_{u_i}^{nc} = - \sum_{s=1}^{n_s} \frac{\phi_{u_i}^{nc} \varphi_{u_s}^{nc} \mathbf{A}^c}{\lambda_{u_i}^{nc} + \lambda_{nc}^{nc} \phi_{u_i}^{nc}} \quad \text{and} \quad \Delta \phi_{\theta_i}^{nc} = - \sum_{s=1}^{n_u} \frac{\phi_{\theta_i}^{nc} \phi_{\theta_s}^{nc} \mathbf{A}^c}{\lambda_{\theta_i}^{nc} + \lambda_{nc}^{nc} \phi_{\theta_i}^{nc}}
$$

This correction leads to an improvement of the uncoupled basis, indicated with superscript $\tilde{nc}$, and gives:

$$
\Delta \tilde{\phi}_{nc}^{\tilde{\phi}} = \begin{bmatrix} \Delta \tilde{\phi}_{nc}^{\tilde{u}} & \Delta \tilde{\phi}_{nc}^{\tilde{\theta}} \end{bmatrix} \begin{bmatrix} \Hat{\eta}_{nc}^{(u)} \\ \Hat{\eta}_{nc}^{(\theta)} \end{bmatrix}
$$

These corrected modes generally show good resemblance with all individual DOF of the state space modes. However, the full corrections would imply operations with all individual uncoupled modes. Therefore it is often not possible to fully develop this correction and in practice one would build the correction with the modes used that are also used in the truncation basis and therefore would give an approximated correction.

### 2.2.5 Method 3b: truncation of a simplified basis of corrected uncoupled modes

Under the special circumstances there is a possibility to build corrections with the full set of modes involved. When the mechanical and thermal spectra are well separated we can recognize that either $\lambda_{u_i}$ or $\lambda_{\theta_i}$ will be dominant, whereas they both appear in the denominator of the correction term of the uncoupled bases. When the spectra of the uncoupled physical fields are separated we can therefore apply a simplification, see ??.

The simplified corrected basis, indicated with superscript $\tilde{nc}$, is given by:

$$
\Delta \tilde{\phi}_{nc}^{\tilde{\phi}} = \begin{bmatrix} \Delta \tilde{\phi}_{nc}^{\tilde{u}} & \Delta \tilde{\phi}_{nc}^{\tilde{\theta}} \end{bmatrix} \begin{bmatrix} \Hat{\eta}_{nc}^{(u)} \\ \Hat{\eta}_{nc}^{(\theta)} \end{bmatrix}
$$

The simplification gives rise to the ability to express the summation in the correction as inverse stiffness or inertia matrices of the uncoupled physical fields. This will be expressed for the two possible extremes of separations of the spectra.

Fast mechanical response compared to thermal response

For this situation we expect $|| \lambda_u ||$ to be much larger than $|| \lambda_\theta ||$ and the correction to the uncoupled bases can be approximated with:

$$
\Delta \phi_{u_i}^{nc} \approx - E_{\theta\theta}^{-1} A_{\theta u} \frac{1}{\lambda_{nc}^{nc}} \phi_{u_i}^{nc} \quad \text{and} \quad \Delta \phi_{\theta_i}^{nc} \approx - A_{\theta u}^{-1} A_{\theta \theta} \phi_{\theta_i}^{nc}
$$

And the simplified corrected basis is given by:

$$
\Delta \tilde{\phi}_{nc}^{\tilde{\phi}} = \begin{bmatrix} -\lambda_{nc}^{-1} E_{\theta\theta}^{-1} A_{\theta u} \phi_{u_i}^{nc} & -A_{\theta u}^{-1} A_{\theta \theta} \phi_{\theta_i}^{nc} \end{bmatrix} \begin{bmatrix} \Hat{\eta}_{nc}^{(u)} \\ \Hat{\eta}_{nc}^{(\theta)} \end{bmatrix}
$$

Slow mechanical response compared to thermal response

For this situation we expect $|| \lambda_\theta ||$ to be much larger than $|| \lambda_u ||$ and the correction to the uncoupled bases can be approximated with:

$$
\Delta \phi_{u_i}^{nc} \approx - A_{\theta \theta}^{-1} A_{\theta u} \phi_{u_i}^{nc} \quad \text{and} \quad \Delta \phi_{\theta_i}^{nc} \approx - \frac{1}{\lambda_{nc}^{nc}} E_{\theta\theta}^{-1} A_{\theta \theta} \phi_{\theta_i}^{nc}
$$

And the simplified corrected basis, again indicated with superscript $\tilde{nc}$, leads to:

$$
\Delta \tilde{\phi}_{nc}^{\tilde{\phi}} = \begin{bmatrix} -A_{\theta \theta}^{-1} A_{\theta u} \phi_{u_i}^{nc} & -\lambda_{nc}^{-1} E_{\theta\theta}^{-1} A_{\theta \theta} \phi_{\theta_i}^{nc} \end{bmatrix} \begin{bmatrix} \Hat{\eta}_{nc}^{(u)} \\ \Hat{\eta}_{nc}^{(\theta)} \end{bmatrix}
$$
3 THE IDEA OF STATIC CORRECTION

In this section we explain the principle of static correction of modal truncation basis for a general second order problem. In section 4 we will perform this correction specifically to the thermomechanical problem.

3.1 Static relation between response and excitation for a second order problem

Suppose we have a system whose dynamics is prescribed by a generalized set of equations such as was written in equation 3. The static solution for this system can be easily obtained from:

\[
\begin{align*}
S &= Bi \\
q &= K^{-1}S \\
y &= L^Tq
\end{align*}
\]

Note that the same static solution can be found from the static relation between response and excitation, that is the static gain of the frequency response function of this system, which can be expressed as:

\[
H_{\text{static}} = L^T K^{-1} S
\]

3.2 Static residualized reduction of a second order problem

In previous analyzes we mentioned that the transfer function is a frequency response function. In order to calculate it, for each frequency the value for transfer function needs to be evaluated, which explains the urge to truncate the system after a limited amount of modes. A relatively cheap extension to the modal truncation given by equation 8 is to statically account for the neglected modes instead of just deleting them:

\[
H \approx \sum_{i=1}^{k} \frac{L_i \psi_i \lambda_i B}{\lambda_i^2} + \sum_{i=k+1}^{n} \frac{L_i \psi_i \lambda_i B}{\lambda_i^2} - j2\zeta\omega - \omega^2
\]

The static residual \( r \) for the truncated modes can easily be recognized to be:

\[
r = \sum_{i=k+1}^{n} \frac{L_i \psi_i \lambda_i B}{\lambda_i^2}
\]

From the latter expression we recognize that the correction accounts for the static solution, as seen from \( K^{-1} \), of the forces that arise from the inertia or acceleration force, seen from \( M \), of the truncated modes. The correction is inconvenient to calculate because it requires all modes to be available. Combining the equations enables to express this static correction of inertia forces contributed by the higher order modes as:

\[
r = L^T K^{-1} B - L^T K^{-1} \sum_{i=1}^{k} M \psi_i \psi_i^T B
\]

In this result we used the residual flexibility \( G \), calculated as

\[
G = K^{-1} \left( I - \sum_{i=1}^{k} M \psi_i \psi_i^T \right)
\]
in order to express a term that represents the inverse of the stiffness matrix after deflation with the modes in the
truncation basis. The obtained static residual can be applied to calculate the correct static result with the reduced
order model. A further extension was proposed, see for example [3] or [4], to use this static correction also as
a vector in the reduction basis. This approach is known as the Modal Truncation Augmentation. Using 28 as an
a posteriori correction for the reduced solution or as a enrichment of the basis guarantees correct static results
calculated with the reduced order model.

3.3 Static residualized reduction of a first order system

Analogous to the derivations in the previous section applied to a general second order problem, we can also do static
correction for problems written in first order form, such as the descriptor formulation. This generalized description
assumes that the system dynamics is prescribed by a generalized set of DOF’s given by equation 4. The static
solution for this system is:

\[
\begin{aligned}
S_{ss} &= B_{ss}^i \\
x &= -A^{-1}S_{ss} \\
y &= L_{ss}^i x
\end{aligned}
\]

and

\[H_{\text{static}} = -L_{ss}^t A^{-1} B_{ss}\]  (30)

Again the same static solution can be found from the static gain of the frequency response function of this system
and can be expressed as:

\[H_{\text{static}} = -\sum_{i=1}^{n} \frac{L_{ss}^i \phi_i^t B_{ss}}{\lambda_i} \]

\[= -L_{ss}^t A^{-1} \sum_{i=1}^{n} E_{\phi_i^t B_{ss}}\]  (31)

A relatively cheap extension to the modal truncation given by equation 11 is again to statically account for the
neglected modes instead of just truncating them:

\[H = \sum_{i=1}^{k} \frac{L_{ss}^i \phi_i^t B_{ss}}{-\lambda_i + i\omega} + \sum_{i=k+1}^{n} \frac{L_{ss}^i \phi_i^t B_{ss}}{-\lambda_i}\]  (32)

The static correction for the truncated modes is calculated from the static residual \(r\) of the modal truncation and can
be recognized to be:

\[r = \sum_{i=k+1}^{n} \frac{L_{ss}^i \phi_i^t B_{ss}}{-\lambda_i} = -L_{ss}^t A^{-1} \sum_{i=k+1}^{n} E_{\phi_i^t B_{ss}}\]  (33)

This static residual can be calculated more conveniently by combining equation 33 and equation 31, which gives:

\[r = -L_{ss}^t A^{-1} \left(I - \sum_{i=1}^{k} E_{\phi_i^t B_s^i}\right) B_{ss}\]  (34)

4 STATIC CORRECTION APPLIED TO DIFFERENT THERMOMECHANICAL REDUCTION BASES

In section 3 the idea of static correction was explained for general systems. In this section we will look how the static
correction can be applied to thermomechanical systems for which the reduction bases were introduced in section 2.
Before the reduced models are manipulated such that they give the correct static solution, we will first discuss this
correct static solution.
4.1 The correct analytical static result

4.1.1 The correct analytical static result for the second order thermomechanical problem

The thermomechanical problem given by equation 1 is described in second order form. The static solution can be easily calculated from:

\[
\begin{bmatrix}
    u \\
    \theta
\end{bmatrix} = \begin{bmatrix}
    K_{uu} & K_{u\theta} \\
    K_{u\theta} & K_{\theta\theta}
\end{bmatrix}^{-1} \begin{bmatrix}
    F \\
    Q
\end{bmatrix}
\] (35)

Because the stiffness matrix is weakly coupled (one-way coupling) we can solve the static solution sequentially. First calculate the static solution for the temperature \( \theta \) as:

\[
\begin{align*}
\theta_{\text{static}} &= K_{\theta\theta}^{-1} Q \\
\end{align*}
\] (36)

The static solution for the displacement can be calculated next, where we substitute the result from equation 36.

This leads to:

\[
\begin{align*}
    u_{\text{static}} &= K_{uu}^{-1} (F - K_{u\theta} \theta) \\
                &= K_{uu}^{-1} F - K_{uu}^{-1} K_{u\theta} K_{\theta\theta}^{-1} Q
\end{align*}
\] (37)

We can recognize both a static displacement due to the applied force \( F \) as a static displacement due to the applied heat \( Q \). The total static result for the set of DOF can be found to be:

\[
\begin{bmatrix}
    u \\
    \theta
\end{bmatrix}_{\text{static}} = \begin{bmatrix}
    K_{uu}^{-1} F - K_{uu}^{-1} K_{u\theta} K_{\theta\theta}^{-1} Q \\
    0
\end{bmatrix}
\] (38)

4.1.2 The correct analytical static result for the first order thermomechanical problem

The derivation of most of the reduction bases is performed when the coupled problem is expressed in first order form. The thermomechanical problem is described in first order form by equation 5. Now we can look whether this gives the same static result as we found for the problem expressed in second order form. The static result is calculated as:

\[
\begin{bmatrix}
    u \\
    \dot{\theta}
\end{bmatrix}_{\text{static}} = \begin{bmatrix}
    K_{uu}^{-1} F - K_{uu}^{-1} K_{u\theta} K_{\theta\theta}^{-1} Q \\
    \frac{1}{\gamma_0} Q
\end{bmatrix}
\] (39)

The static system matrix \( A \) expresses the coupling of the entire vector of state variables \( x \). Because this vector also contains the mechanical velocity \( \dot{u} \), the coupling from mechanical velocity to the thermal field is also expressed in \( A \). This again implies that the matrix now contains two-sided coupling and we cannot perform the sequential approach such as used before. However, the result from 39 can be compared to the static result obtained from the second order system, because \( x \) and \( \theta \) are identical to the results from equation 37 and equation 36 respectively.

In equation 39 we find that statically \( \dot{x} = 0 \) is calculated, which is the expected result.

4.2 The static residual for different thermomechanical reduction methods

4.2.1 The static residual for method 1: fully coupled modes

For reduction method 1 we used a basis that consists of the fully coupled modes calculated from the first order system. Therefore we can directly apply equation 31 to calculate the static residual, indicated with \( r \), which gives:

\[
r = -A^{-1} \left( I - \sum_{i=1}^{k} E \phi_i \phi_i^T \right) \begin{bmatrix}
    \dot{F} \\
    \dot{Q}
\end{bmatrix}
\] (40)
Note that only the fully coupled modes are correct modes for the coupled problem. Therefore they obtain the original orthogonality properties with respect to the system matrices $A$ and $E$. Other reduction basis may give a sufficient representation of the coupled problem, but are not correct modes for the coupled problem. Therefore they do not preserve the orthogonality properties. However, often these methods do contain orthogonality properties with submatrices of the coupled problem and this needs to be fully exploited when calculating the static correction.

### 4.2.2 The static residual for method 2a: fully uncoupled modes of the second order problem

With this method we use a basis of uncoupled modes of the second order problem. The static correction cannot be calculated from equation 26 (re-written for an unsymmetric problem), because as stated above the basis does not consist of correct modes of the coupled problem. The modes in the bases do have orthogonality properties with respect to the system matrices that can be identified to belong to the uncoupled problems. Alternatively however, because equation 26 cannot be applied, we can perform a sequential approach such as used to obtain the analytical static solution before. This means that we can express the static residual in terms of the contribution of the individual DOF corresponding to the separate physical fields. Recall that in method 2a we used the basis given by equation 15, from which we find that the reduced order model writes a static relation $\hat{K}\hat{z} = \hat{F}$ that is also expressed in a one-way coupled form as:

$$
\begin{bmatrix}
\hat{\Psi}^{(u)} K_{uu} \hat{\Psi}^{(u)} \\
\hat{\Psi}^{(u)} \hat{K}_{u\theta} \hat{\Psi}^{(\theta)} \\
\hat{\Psi}^{(\theta)} K_{\theta\theta} \hat{\Psi}^{(\theta)}
\end{bmatrix}
\begin{bmatrix}
\hat{z}^{(u)} \\
\hat{z}^{(\theta)}
\end{bmatrix}
= 
\begin{bmatrix}
\hat{\Psi}^{(u)^T} F \\
\hat{\Psi}^{(\theta)^T} Q
\end{bmatrix}
$$

(41)

Following the sequential approach we first look for the (incorrect) static solution for temperature $\theta$ that the reduced order model gives. Note that we can use the orthogonality of $\hat{\Psi}^{(\theta)}$ with respect to $K_{\theta\theta}$. This gives:

$$
\hat{q}_{stat}^{\theta} = \hat{\Psi}^{(\theta)} \left( \hat{\Psi}^{(\theta)} K_{\theta\theta}^{-1} \hat{\Psi}^{(\theta)} \right)^{-1} \hat{\Psi}^{(\theta)^T} Q = \sum_{i=1}^{k_{\theta}} \frac{\phi_i^{(\theta)} \phi_i^{(\theta)^T}}{\lambda_{\theta i}} Q
$$

(42)

A static error is recognized because, compared to the correct result in equation 36, only part of the spectral expansion of $K_{\theta\theta}^{-1}$ is performed. From this result we see that the static residual is identical to that of a truncated transfer function of a first order system as is given by equation 34. The static residual can therefore be expressed with use of a thermal residual flexibility matrix $G_{\theta\theta}$ as:

$$
\hat{r}_{\theta} Q_{\theta} = G_{\theta\theta} Q
$$

(43)

We will look at the mechanical static residual for $u$ next. From equation 38 we recognized that the correct static solution $u$ consist of a contribution by $F$ and a contribution by $Q$ and therefore we can also expect a static residual for both contributions and for convenience we will treat them separately. The (incorrect) static mechanical solution due to an applied force $F$ calculated with the reduced model is:

$$
\hat{u}_{stat}^{F} = \hat{\Psi}^{(u)} \left( \hat{\Psi}^{(u)} K_{uu}^{-1} \hat{\Psi}^{(u)} \right)^{-1} \hat{\Psi}^{(u)^T} F = \sum_{i=1}^{k_u} \frac{\phi_i^{(u)} \phi_i^{(u)^T}}{\lambda_{u i}} F
$$

(44)

Compared to the correct static solution as seen in equation 37, we see that the spectral expansion of $K_{uu}^{-1}$ is not complete and therefore this part of the static residual is identical to that of a truncated transfer function as is given by equation 28. The static residual is expressed using the mechanical residual flexibility matrix $G_{uu}$ as:

$$
\hat{r}_{F_{\theta}} = G_{uu} F
$$

(45)

The (incorrect) static mechanical solution due to an applied force $Q$ can be calculated as follows:

$$
\hat{u}_{stat}^{Q} = -\hat{\Psi}^{(u)} \hat{K}_{uu}^{-1} \hat{K}_{u\theta} \hat{\theta}_{stat}^{\theta}
= -\hat{\Psi}^{(u)} \left( \hat{\Psi}^{(u)^T} K_{uu}^{-1} \hat{\Psi}^{(u)} \right)^{-1} \hat{\Psi}^{(u)^T} K_{u\theta} \hat{\theta}_{stat}^{\theta}
$$

(46)

Compared to the correct result found in equation 37 we see that 2 effects will introduce a static error. The first source is due to the incomplete spectral expansion of $K_{uu}^{-1}$. The second type of error arises from the fact that $\hat{\theta}_{stat}^{\theta}$ is used
instead of \( \theta_{\text{static}} \), where from equation 42 we already found this solution contains the static residual expressed in equation 43. This first source introduces a static residual containing the mechanical residual flexibility:

\[
 r_{Q \rightarrow u; \text{source 1}} = G_{uu} K_{u0} \theta_{\text{static}} = -G_{uu} K_{u0} K_{\theta\theta}^{-1} Q \tag{47}
\]

The second source introduces a following static residual containing the thermal residual flexibility:

\[
 r_{Q \rightarrow u; \text{source 2}} = K_{u0}^{-1} K_{u\theta} r_\theta = -K_{u0}^{-1} K_{u0} G_{\theta\theta} Q \tag{48}
\]

Altogether we can recognize the 4 types of contributions to a static residual. All of them are related to the fact that the truncated basis leads to incomplete spectral expansion of the stiffness matrices. The results are summarized in 1. Again all these partial corrections can be used to correct \textit{a posteriori} the solution obtained from the truncated basis, or to enrich the truncated basis.

### 4.2.3 The static residual for method 2b: fully uncoupled modes of the first order system problem

Comparable to the method described just before this method uses a basis of uncoupled modes. However for this method these are calculated from a first order problem. Again the static correction cannot be calculated from equation 31, because the basis does not consist of correct modes of the coupled first order problem, but the uncoupled modes of this first order problem contain similar information as those of the second order problem. Because also the correct static solution of both the second and the first order system are very alike we expect to find a static residual that is also much alike.

Before we calculate the (incorrect) static result of the reduced order model by hand, we especially look on what the reduced order static relation looks like. Therefore we first discuss the mechanical uncoupled basis. Because of the thermal problem is a first order problem we see that \( \Phi \) belonging to different eigenfrequencies and then their complex conjugates:

\[
 \Phi^{(u)} = \begin{bmatrix} \Psi^{(u)} & \Psi^{(u)} \\ \Psi^{(u)} A_u & -\Psi^{(u)} A_u \end{bmatrix} \Rightarrow \Phi = \begin{bmatrix} \Psi^{(u)} & \Psi^{(u)} \\ \Psi^{(u)} A_u & -\Psi^{(u)} A_u \end{bmatrix}
\]

\[
 \Phi^{(u)} = \begin{bmatrix} \Psi^{(u)} & \Psi^{(u)} \\ \Psi^{(u)} A_u & -\Psi^{(u)} A_u \end{bmatrix} \Rightarrow \Phi = \begin{bmatrix} \Psi^{(u)} & \Psi^{(u)} \\ \Psi^{(u)} A_u & -\Psi^{(u)} A_u \end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Indication</th>
<th>Expression</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_{Q \rightarrow \theta} )</td>
<td>( K_{\theta\theta} \left { I - \sum_{i=1}^{k_u} C_{\theta\theta} \psi_i^{(\theta)} \psi_i^{(\theta)^T} \right } Q )</td>
<td>Error in ( \theta ) as a result of ( Q ), due to incomplete expansion of ( K_{\theta\theta}^{-1} )</td>
</tr>
<tr>
<td>( r_{F \rightarrow u} )</td>
<td>( K_{uu} \left { I - \sum_{i=1}^{k_u} M_{uu} \psi_i^{(u)} \psi_i^{(u)^T} \right } F )</td>
<td>Error in ( u ) as a result of ( F ), due to incomplete expansion of ( K_{uu}^{-1} )</td>
</tr>
<tr>
<td>( r_{Q \rightarrow u; \text{source 1}} )</td>
<td>( -K_{uu}^{-1} \left { I - \sum_{i=1}^{k_u} M_{uu} \psi_i^{(u)} \psi_i^{(u)^T} \right } K_{u0} K_{\theta\theta} Q )</td>
<td>Error in ( u ) as a result of ( Q ), due to incomplete expansion of ( K_{uu}^{-1} )</td>
</tr>
<tr>
<td>( r_{Q \rightarrow u; \text{source 2}} )</td>
<td>( -K_{uu}^{-1} K_{u0} K_{\theta\theta} \left { I - \sum_{i=1}^{k_u} C_{\theta\theta} \psi_i^{(\theta)} \psi_i^{(\theta)^T} \right } Q )</td>
<td>Error in ( u ) as a result of ( Q ), due to incomplete expansion of ( K_{\theta\theta}^{-1} )</td>
</tr>
</tbody>
</table>

**TABLE 1**: Table of expected residuals for a basis of uncoupled modes of the second order problem.
The static relation \(Az = S\) expressed with this basis is then given as:

\[
\begin{bmatrix}
2\Lambda_u \Psi^t_u K_{uu} \Psi_u \\
\Lambda_u \Psi^t_u K^t_{u\theta} \Psi_u \\
\Lambda_u \Psi^t_{u\theta} K_{u\theta} \Psi_u \\
\Lambda_{u\theta} \Psi^t_{u\theta} K^t_{u\theta} \Psi_u \\
\Lambda_{u\theta} \Psi^t_{u\theta} K_{u\theta} \Psi_{u\theta}
\end{bmatrix}
\begin{bmatrix}
\Lambda_u \Psi^t_u K_{u\theta} \Psi_u \\
\Lambda_u \Psi^t_{u\theta} K^t_{u\theta} \Psi_u \\
\Lambda_u \Psi^t_{u\theta} K_{u\theta} \Psi_{u\theta}
\end{bmatrix}
\begin{bmatrix}
\tilde{z}_{u1} \\
\tilde{z}_{u2} \\
\tilde{z}_{\theta}
\end{bmatrix}
= \begin{bmatrix}
\Lambda_u \Psi^t_u F \\
\Lambda_u \Psi^t_{u\theta} F \\
\frac{1}{T_u} \Psi^t_{\theta} Q
\end{bmatrix}
\tag{51}
\]

By inspection of this static relation we recognize that the first and second line express the same operation and therefore the correct solution gives \(z_u = z_{u2}\). If one implements this solution in the third line, we immediately find that the contribution of \(\Lambda_u \Psi^t_{u\theta} K_{u\theta} \Psi_u\) and \(-\Lambda_u \Psi^t_{u\theta} K_{u\theta} \Psi_{u\theta}\) will cancel each other, resulting to a one-way coupled problem again. Note that the solution for the individual DOF can be written as:

\[
\begin{bmatrix}
u \\
\dot{\nu} \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
\Psi^{(u)}(u) & \Psi^{(u)}(u) & \Psi^{(u)}(u) \\
\Psi^{(u)}(u) & -\Psi^{(u)}(u) \Lambda_u & \Psi^{(u)}(u) \\
& & \Psi^{(u)}(u)
\end{bmatrix}
\begin{bmatrix}
\tilde{z}_{u1} \\
\tilde{z}_{u2} \\
\tilde{z}_{\theta}
\end{bmatrix}
\tag{52}
\]

If we now use \(z_{u1} = z_{u2}\), we see that indeed the solution for \(\dot{\nu} = 0\) is found. Because the static relation can be interpreted to be one-way coupled now, we can find the static solution for \(z_{u1}\) and \(z_{\theta}\) with a sequential solution procedure to be:

\[
\begin{align*}
\tilde{z}_{\theta} &= \left(\Psi^{(\theta)^t} K_{\theta\theta} \Psi^{(\theta)}\right)^{-1} \Psi^{(\theta)^t} Q \\
z_{u1} = z_{u2} &= \frac{1}{2} \left(\Psi^{(u)^t} K_{uu} \Psi^{(u)}\right)^{-1} \left(\Psi^{(u)^t} F - \Psi^{(u)^t} K_{u\theta} \Psi^{(\theta)} \tilde{z}_{\theta}\right)
\end{align*}
\tag{53}
\]

The static solution is given in a form that we have seen several times before. We can now reason that if we use a truncated basis, but guarantee that we keep the modes in complex conjugate pairs, we can again recognize two identical operations that result to \(\tilde{z}_{u1} = \tilde{z}_{u2}\). This leads to a one-way coupling in the static relation of the reduced order model. As one can verify now we will obtain the same static result as with a basis of uncoupled second order modes. This also induces that we will find the same static residuals for \(u\) and \(\theta\) as were summarized in table 1. The residual for \(\dot{\nu} = 0\) means that we indeed guaranteed that similar to equations ?? and ?? some terms dropped out. The interpretation of this from a reduction point of view is that we neatly represented the identity relation that was used to write a second order problem into first order form.

Now let’s return to the general case that we do not intentionally put modes in conjugate pairs in the basis. The resulting (incorrect) static solution can then be found to be:

\[
\begin{bmatrix}
\dot{\tilde{z}} \\
\tilde{z}_{u1} \\
\tilde{z}_{\theta}
\end{bmatrix} = \begin{bmatrix}
\tilde{\Psi}^{(u)} \left(K_{uu} - \tilde{\Lambda}_u \tilde{K}_{u\theta} \tilde{K}^{-1}_{u\theta} \tilde{K}^t_{u\theta}\right)^{-1} \left(\tilde{F} - \tilde{K}_{u\theta} \tilde{K}^{-1}_{u\theta} \tilde{Q}\right) \\
\tilde{\Lambda}_u \tilde{\Psi}^{(u)} \left(\tilde{K}_{uu} - \tilde{\Lambda}_u \tilde{K}_{u\theta} \tilde{K}^{-1}_{u\theta} \tilde{K}^t_{u\theta}\right)^{-1} \left(\tilde{F} - \tilde{K}_{u\theta} \tilde{K}^{-1}_{u\theta} \tilde{Q}\right) \\
\tilde{\Psi}^{(\theta)} \left(\tilde{K}_{\theta\theta} - \frac{1}{2} \tilde{\Lambda}_u \tilde{K}_{\theta\theta} \tilde{K}^{-1}_{u\theta} \tilde{K}^t_{u\theta}\right)^{-1} \left(\tilde{Q} - \frac{1}{2} \tilde{\Lambda}_u \tilde{K}_{\theta\theta} \tilde{K}^{-1}_{u\theta} \tilde{F}\right)
\end{bmatrix}
\tag{54}
\]

From this result we can see that due to incomplete conjugate sets of modes, several errors are introduced that we know that should not be there, such as a static solution for \(\dot{\tilde{z}}\) arising from both \(\tilde{F}\) and \(\tilde{Q}\), a contribution of \(\tilde{F}\) in the static solution for \(\tilde{\theta}\) and also we see several phase errors that are introduced. With a lot of effort we were able to write these residuals analytically, but in practice we will not use such analytical forms. Therefore we do not write these, but suggest to avoid many of these unnecessary errors by always putting the modes in complex conjugate pairs.

4.3 The static residual for method 3a and method 3b: corrected uncoupled modes of the first order problem

In this section we will look at the reduced static solution for bases in which a correction term was introduced. We can immediately recognize one disadvantage, i.e. the fact that the bases do not consist of block diagonal matrices, but also contain off-diagonal terms. According to equation 18, 21 or 23 the corrected uncoupled basis looks like:

\[
\tilde{\Phi}^{\tilde{nc}} = \begin{bmatrix}
\Phi^{(u)} & \Delta \Phi^{(u)} \\
\Delta \Phi^{(u)} & \tilde{\Phi}^{(\theta)}
\end{bmatrix}
= \begin{bmatrix}
\tilde{\Phi}_{u} & \tilde{\Phi}_{\theta}
\end{bmatrix}
\tag{55}
\]
The absence of a block-diagonal form leads to a fully coupled reduced static relation, which means in other words that the correction modifies the submatrices that we recognized to belong to the uncoupled physical problems, as was probably the intention of the correction. This means however that if one now wants to guarantee the correct static solution, this becomes very intricate because the correction is not derived from an eigenvalue problem. The fully coupled reduced static relation is given by:

\[
\begin{bmatrix}
\Phi^{uc}_{\theta} A \Phi^{uc}_{\theta} & \Phi^{uc}_{\theta} A \Phi^{uc}_{\theta} \\
\Phi^{uc}_{u} A \Phi^{uc}_{u} & \Phi^{uc}_{u} A \Phi^{uc}_{u}
\end{bmatrix}
\begin{bmatrix}
z_u \\
z_\theta
\end{bmatrix}
= \begin{bmatrix}
\Phi^{uc}_{\theta} F \\
\Phi^{uc}_{\theta} Q
\end{bmatrix}
\tag{56}
\]

and the results for the individual DOF are represented as:

\[
\hat{u} = \Phi_u z_u + \Delta \hat{\phi}_u z_\theta \quad \text{and:} \quad \hat{\theta} = \Delta \hat{\phi}_u z_u + \hat{\phi}_u z_\theta
\tag{57}
\]

We recognize that the results for the DOF depend on the modal amplitudes corresponding to both fields at the same time, as was indeed the intention of the correction and the approximate static result is now obtained from:

\[
\hat{u} = \Phi_u \left( \bar{A}_{uu} - \bar{A}_{u\theta} \bar{A}_{\theta u} \right)^{-1} \left( \hat{S}^{(u)} - \bar{A}_{u\theta} \bar{A}_{\theta u} \hat{S}^{(\theta)} \right) + \ldots
\]

\[
\Delta \hat{\phi}_u \left( \bar{A}_{\theta u} - \bar{A}_{u\theta} \bar{A}_{\theta u} \right)^{-1} \left( \hat{S}^{(\theta)} - \bar{A}_{u\theta} \bar{A}_{\theta u} \hat{S}^{(u)} \right) + \ldots
\]

\[
\hat{\theta} = \Delta \hat{\phi}_u \left( \bar{A}_{uu} - \bar{A}_{u\theta} \bar{A}_{\theta u} \right)^{-1} \left( \hat{S}^{(u)} - \bar{A}_{u\theta} \bar{A}_{\theta u} \hat{S}^{(\theta)} \right) + \ldots
\]

\[
\Delta \hat{\phi}_u \left( \bar{A}_{\theta u} - \bar{A}_{u\theta} \bar{A}_{\theta u} \right)^{-1} \left( \hat{S}^{(\theta)} - \bar{A}_{u\theta} \bar{A}_{\theta u} \hat{S}^{(u)} \right)
\tag{58}
\]

where the individual terms can be read from appendix A. It is difficult to foresee the influence of the correction terms. There are however a few possibilities that can be used for static residual. A first possibility is to enforce that the corrections are orthogonal to the mono-physical modes. This simplifies the calculation a lot and for example such as written in the work by Tournour, see [5], the pseudostatic contribution of truncated modes can be taken into account. In order to do this the solution for \( u \) and \( \theta \) such as given by equation 1 are expressed for a harmonic system as:

\[
(K_{uu} + i\omega C_{uu} - \omega^2 M_{uu}) u = (F - K_{u\theta} \theta)
\]

\[
(K_{\theta\theta} + i\omega C_{\theta\theta}) \theta = (Q - i\omega K^*_{u\theta} u)
\tag{59}
\]

A so-called pseudostatic solution \( u_0 \) and \( \theta_0 \) can now be written as:

\[
u_0 = K_{uu}^{-1} (F - K_{u\theta} \theta)
\]

\[
\theta_0 = K_{\theta\theta} (Q - i\omega K^*_{u\theta} u)
\tag{60}
\]

The same procedure can be written for the uncoupled modal basis such as expressed in equation 14 and gives:

\[
u_0 = \hat{\Psi}^{(u)} \hat{\Lambda}^{(u)^{-1}} \hat{\Psi}^{(u)} (F - K_{u\theta} \theta)
\]

\[
\theta_0 = \hat{\Psi}^{(\theta)} \hat{\Lambda}^{(\theta)^{-1}} \hat{\Psi}^{(\theta)} (Q - i\omega K^*_{u\theta} u)
\tag{61}
\]

This means that we can express the pseudostatic residual now as:

\[
r_u = \left( K_{uu}^{-1} - \hat{\Psi}^{(u)^*} \hat{\Lambda}^{(u)^{-1}} \hat{\Psi}^{(u)} \right) (F - K_{u\theta} \theta)
\]

\[
r_{\theta} = \left( K_{\theta\theta}^{-1} - \hat{\Psi}^{(\theta)^*} \hat{\Lambda}^{(\theta)^{-1}} \hat{\Psi}^{(\theta)} \right) (Q - i\omega K^*_{u\theta} u)
\tag{62}
\]

If we now implement the modal approximations for \( u \) and \( \theta \) in the righthand side of these residuals, we obtain:

\[
r_u = \left( K_{uu}^{-1} - \hat{\Psi}^{(u)^*} \hat{\Lambda}^{(u)^{-1}} \hat{\Psi}^{(u)} \right) (F - K_{u\theta} \hat{\Psi}^{(u)} z_\theta)
\]

\[
r_{\theta} = \left( K_{\theta\theta}^{-1} - \hat{\Psi}^{(\theta)^*} \hat{\Lambda}^{(\theta)^{-1}} \hat{\Psi}^{(\theta)} \right) (Q - i\omega K^*_{u\theta} \hat{\Psi}^{(u)} z_u)
\tag{63}
\]
Another possibility is to ensure also static correctness of the corrections that were suggested by equation 17. From this equation we noticed that theoretically all modes are needed to calculate the desired correction, whereas in practice we just have a limited amount of modes; namely those that were also used as basis for the reduced models. When the correction is calculated with a truncated set of modes we do not obtain the full correction desired. From section 2.2.5 we recognized however that the full spectral expansion of either the inertia matrix or the stiffness matrix can serve as an approximation of the correction. Therefore we now can suggest at least to use this approximation for the modes that are truncated. Comparable to the residual flexibility given by equation 28 we suggest to introduce a residual correction term for a single mode:

\[
\Delta \phi_{ui}^{nc} \approx - \sum_{s=1}^{k_u} \frac{\phi_{ui,nc}^{s} \phi_{ui}^{nc}}{\lambda_{ui}^{nc} + \lambda_{uy}^{nc}} A_{uu}^{s} \left( I - \sum_{s=1}^{k_u} C_{uy}^{s} \phi_{ui,nc}^{s} \phi_{ui}^{nc} \right) \]

\[
\Delta \phi_{ui}^{yy} \approx - \sum_{s=1}^{k_u} \frac{\phi_{ui,nc}^{s} \phi_{ui}^{nc}}{\lambda_{ui}^{nc} + \lambda_{uy}^{nc}} A_{uu}^{s} \left( I - \sum_{s=1}^{k_u} E_{uy}^{s} \phi_{ui,nc}^{s} \phi_{ui}^{nc} \right) A_{uy}^{s} \]

\[
K_{ui}^{t} \phi_{ui}^{nc} = - \sum_{s=1}^{k_u} \frac{\phi_{ui,nc}^{s} \phi_{ui}^{nc}}{\lambda_{ui}^{nc} + \lambda_{uy}^{nc}} A_{uu}^{s} \left( I - \sum_{s=1}^{k_u} C_{uy}^{s} \phi_{ui,nc}^{s} \phi_{ui}^{nc} \right) K_{ui}^{t} \phi_{ui}^{nc} - G_{uy}^{t} \phi_{ui}^{nc}
\]

5 CONCLUSIONS AND OUTLOOK

In this paper we investigated different reduction bases for thermomechanically coupled problems. The intention of all bases is to have a modal representation of the complete or a specific part of the thermomechanical equation of motion. Although modes are capable to give a dynamic representation of the system they do not give the correct static result. In order to identify this error and have an opportunity to improve the reduced models, we investigated the static residuals introduced when applying these different variants.

The correct static results predicts that an applied heat generates both a static temperature distribution and a mechanical deformation. An applied force only yields a static deformation. Both loads lead to a zero static velocity. When applying the different bases, we observed the following effects. Fully coupled modes give the possibility to express the thermomechanical problem as modal summation. The static residual can easily be implemented by calculation of the static contribution of the truncated modes by calculation of their corresponding modal acceleration.

The static residual when using a basis of uncoupled modes of the second problem could be calculated by using the property of one-sided static coupling, such that the residuals could be calculated sequentially. This approach resulted to static errors due to incomplete spectral expansions of necessary inverse matrices.

Coupled modes of the problem are the solution to the thermomechanical equation written in first order form. In first order form the static relation writes a two-sided coupling between the thermal and the mechanical DOF. Using a basis of uncoupled modes calculated in first order form gave rise to more contributions of static error because of this two-sided coupling. It resulted to unexpected contribution from force to the thermal DOF and undesired static solution for velocity. It was observed that these errors disappear when modes are used complex conjugate pairs. Bases that include corrections on the uncoupled modes introduced off-diagonal terms in the bases by which the reduced matrices contain too much contributions to the static residual to express them analytically. The choice for the specific type of correction possibly diminishes or deteriorates the static results.

In future work we plan to use the terms calculated as static residuals as an enrichment to the bases described in this paper. We suggest to use modes in complex pairs, because this is a simple extension and obviously prevents the occurrence of errors on the static results.

ACKNOWLEDGEMENTS

We acknowledge the MicroNed program of the Ministry of Economic Affairs of the Netherlands for the financial support.
A APPENDIX: TERMS IN THE FULLY COUPLED REDUCED MATRIX

For a basis $\Phi^\sim nc$ the reduced static relation looks like:

$$
\begin{bmatrix}
\Phi^\sim nc^T A \Phi^\sim nc \\
\Phi^\sim \theta^T A \Phi^\sim \theta
\end{bmatrix}
\begin{bmatrix}
\Phi^\sim nc^T \\
\Phi^\sim \theta^T
\end{bmatrix}
\begin{bmatrix}
z_u \\
z_\theta
\end{bmatrix}
=
\begin{bmatrix}
\Phi^\sim nc^T F \\
\Phi^\sim \theta^T Q
\end{bmatrix}
$$

(65)

where individual terms are:

$$
\tilde{A}_{uu} = \hat{\Phi}^{(u)}^T K_{uu} \hat{\Phi}^{(u)} + \hat{\Phi}^{(u)}^T K_{u\theta} \hat{\Phi}^{(\theta)} + \Delta \hat{\Phi}^{(u)} K_{u\theta} \Delta \hat{\Phi}^{(u)} + \Delta \hat{\Phi}^{(u)} K_{u\theta} \Delta \hat{\Phi}^{(\theta)}
$$

$$
\tilde{A}_{u\theta} = \hat{\Phi}^{(u)}^T K_{uu} \Delta \hat{\Phi}^{(\theta)} + \hat{\Phi}^{(u)}^T K_{u\theta} \hat{\Phi}^{(\theta)} + \Delta \hat{\Phi}^{(u)} K_{u\theta} \Delta \hat{\Phi}^{(\theta)} + \Delta \hat{\Phi}^{(u)} K_{u\theta} \Delta \hat{\Phi}^{(\theta)}
$$

$$
\tilde{A}_{\theta u} = \Delta \hat{\Phi}^{(\theta)} K_{uu} \hat{\Phi}^{(u)} + \Delta \hat{\Phi}^{(\theta)} K_{u\theta} \hat{\Phi}^{(u)} + \hat{\Phi}^{(\theta)} K_{u\theta} \Delta \hat{\Phi}^{(u)} + \hat{\Phi}^{(\theta)} K_{u\theta} \Delta \hat{\Phi}^{(u)}
$$

$$
\tilde{A}_{\theta\theta} = \Delta \hat{\Phi}^{(\theta)} K_{uu} \Delta \hat{\Phi}^{(\theta)} + \Delta \hat{\Phi}^{(\theta)} K_{u\theta} \hat{\Phi}^{(\theta)} + \hat{\Phi}^{(\theta)} K_{u\theta} \Delta \hat{\Phi}^{(\theta)} + \hat{\Phi}^{(\theta)} K_{u\theta} \Delta \hat{\Phi}^{(\theta)}
$$

$$
\tilde{S}^{(u)} = \hat{\Phi}^{(u)}^T F + \Delta \hat{\Phi}^{(u)} Q
$$

$$
\tilde{S}^{(\theta)} = \Delta \hat{\Phi}^{(\theta)} F + \hat{\Phi}^{(\theta)} Q
$$

(66)

REFERENCES


