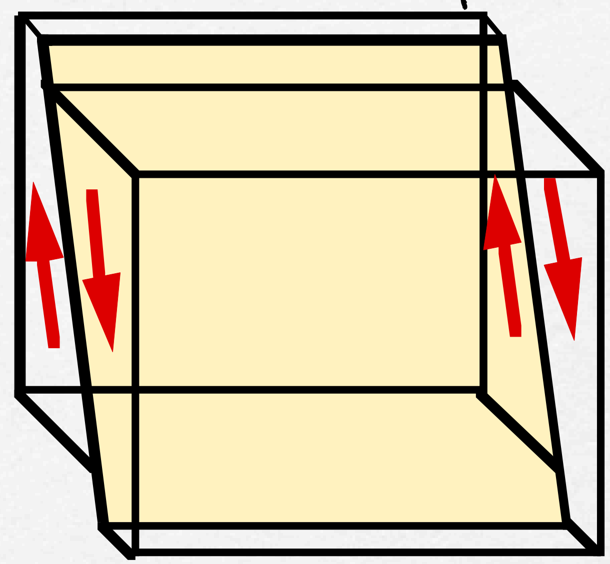
Multiaxial fatigue

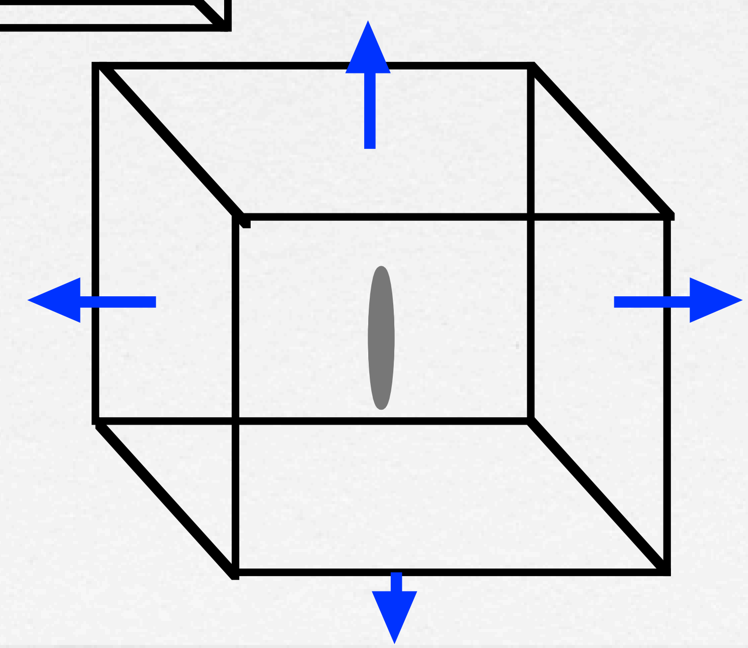
an introduction

Multiaxial fatigue

- Uniaxial fatigue
  - One stress (strain) component
  - Fatigue related to amplitude and mid value of this component
- Multiaxial fatigue
  - Six independent components
  - What is the amplitude and mid value?



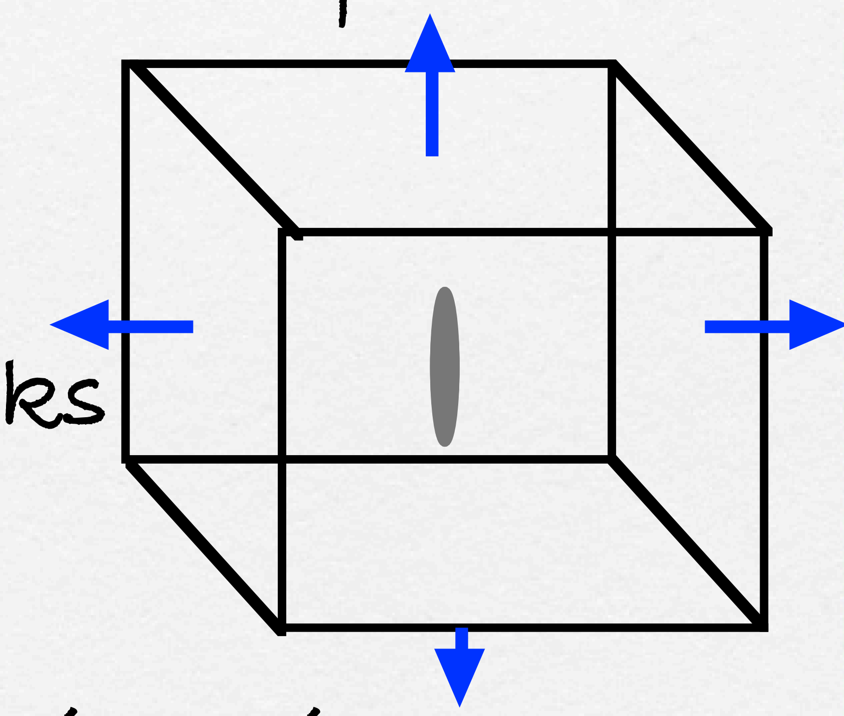
- Assume that, in the general case, fatigue behaviour is influenced by
  - shear stress
  - hydrostatic stress



Hydrostatic stress

- The hydrostatic stress is the mean value of normal stresses acting on the material point (positive in tension)

- A tensile (positive) hydrostatic stress opens up microscopic cracks



$$\sigma_\mathrm{h} = (\sigma_x + \sigma_y + \sigma_z)/3$$

- The hydrostatic stress is a stress invariant

$$\sigma_\mathrm{h} = \sigma_{ii}/3$$

- regardless of coordinate system

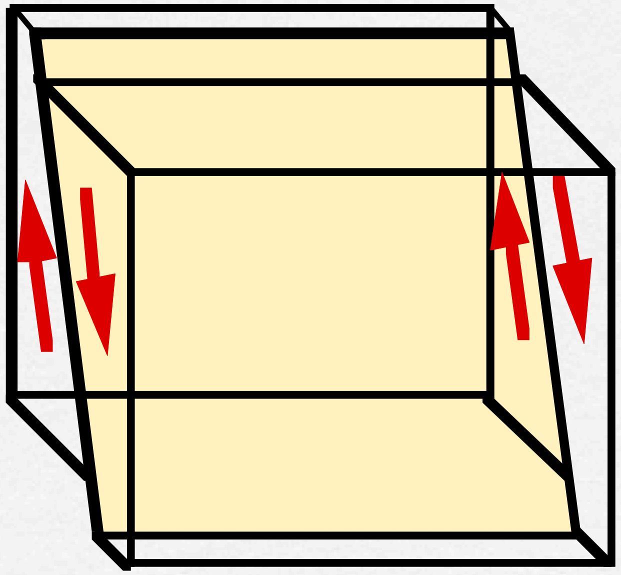
Shear stress measures

- The shear stress initiates slip bands which leads to microscopic cracks
- Since a static shear stress have no influence on the fatigue damage, the shear stress "amplitude" is employed
- Two usual measures
  - Tresca shear stress

$$\tau_{\mathrm{Tr}} = (\sigma_1 - \sigma_3)/2$$

  - von Mises stress

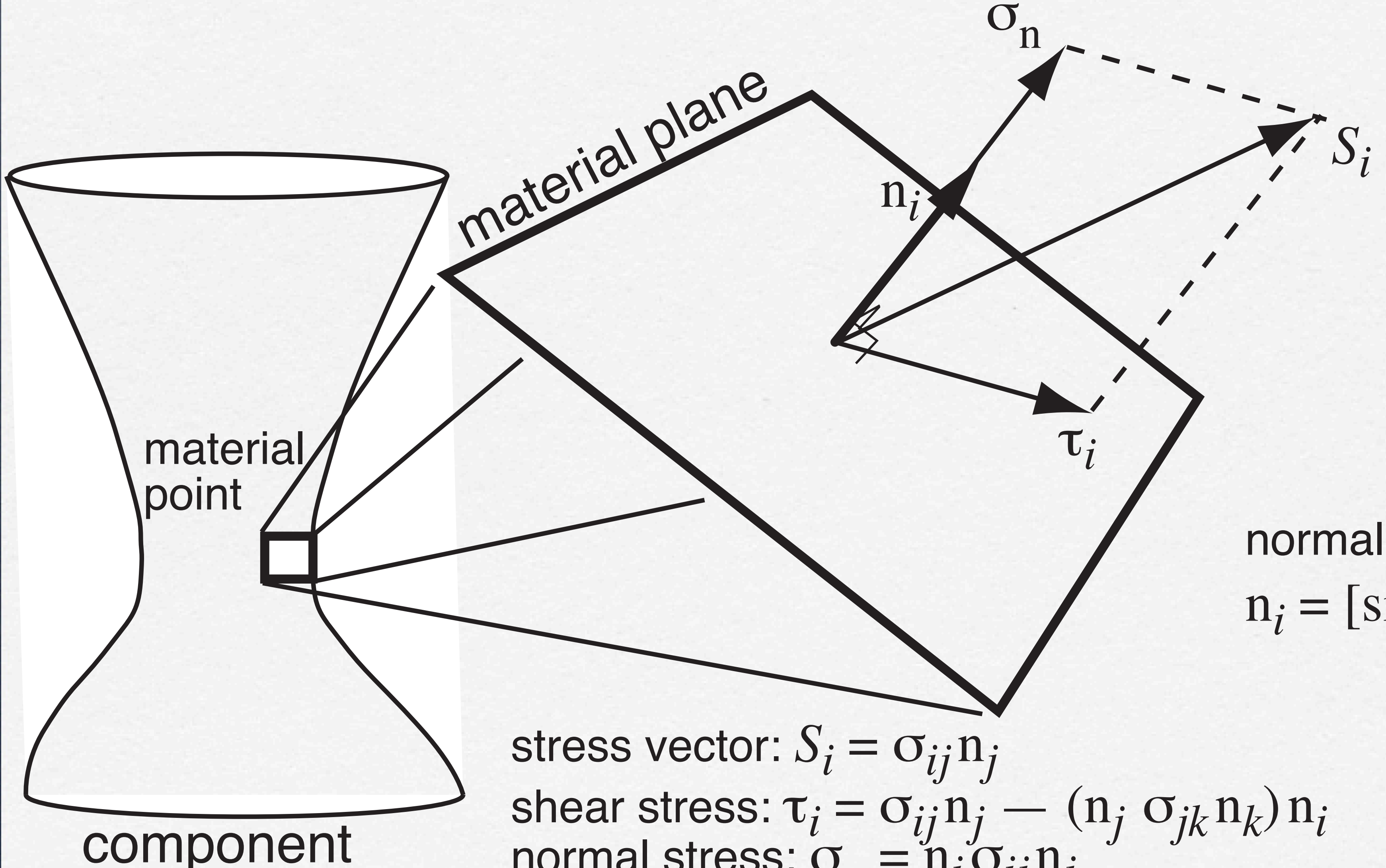
$$\sigma_{\mathrm{vM}} = \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}/\sqrt{2}$$

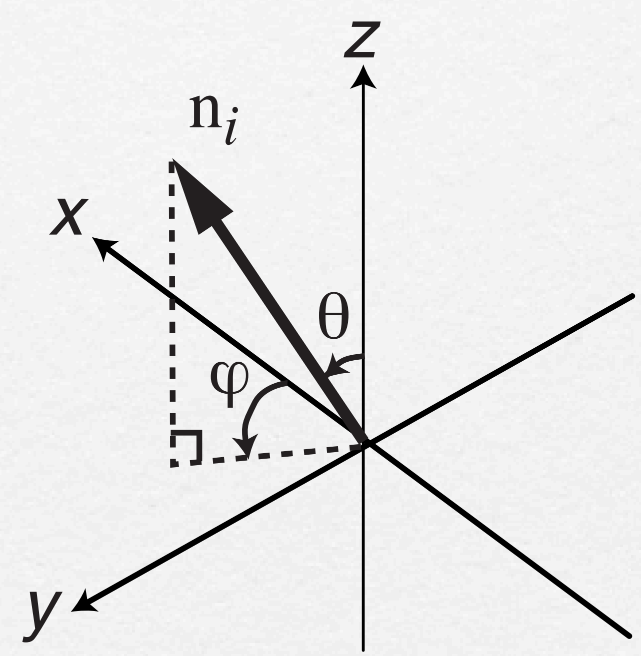


Shear stress "amplitude"

- Empirical experience: superposed static shear stress does not have any influence on the fatigue limit:
  $\tau_{\mathrm{FL}} = \tau_{FLP}$
  whereas
  $\sigma_{\mathrm{FL}} \neq \sigma_{FLP}$
- A shear stress "amplitude" is therefore needed in multiaxial fatigue initiation (HCF) criteria
- This "amplitude" the difference between current shear stress magnitude and the mid value of the shear stress for the current stress cycle

Stresses in a material point





normal vector:

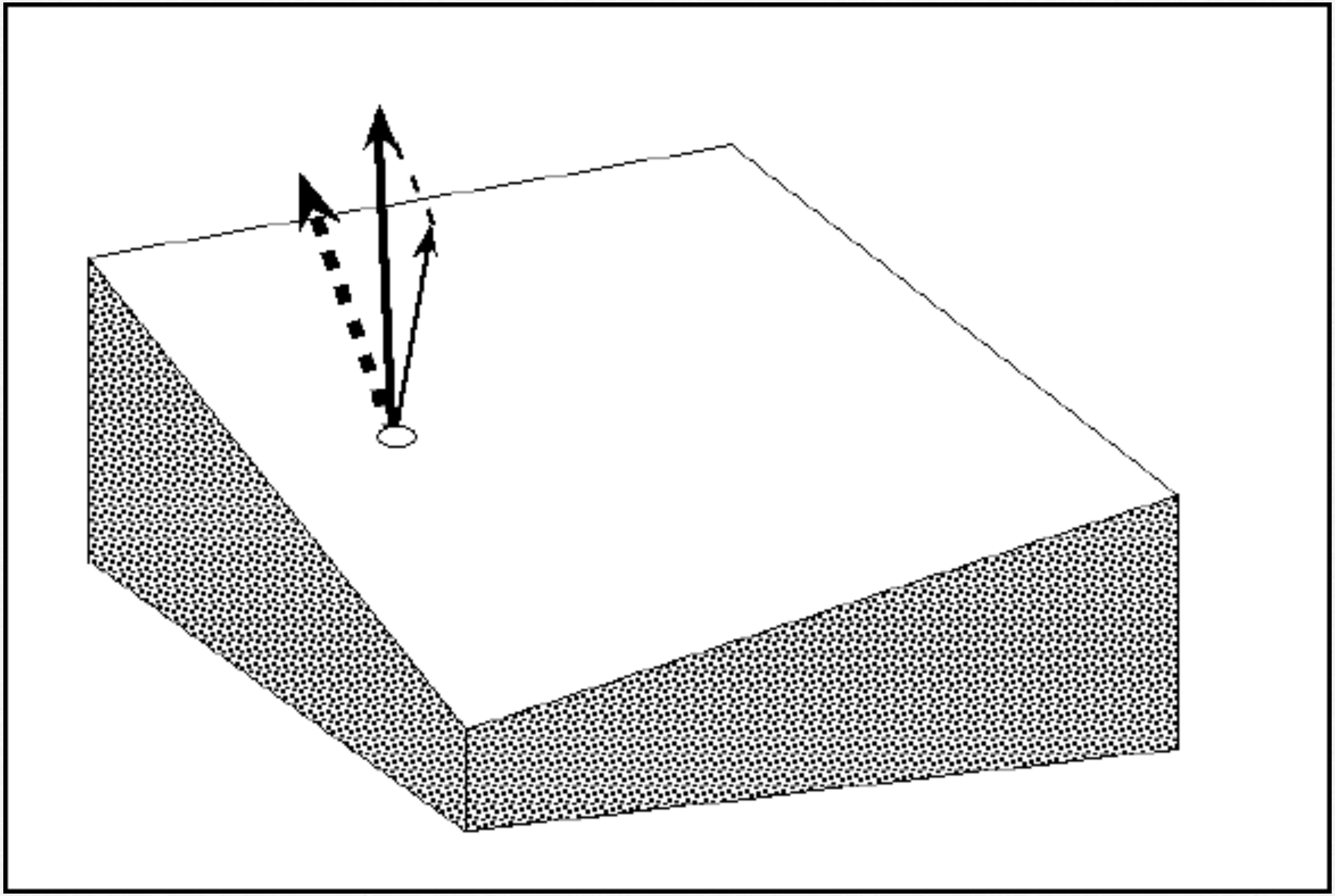
$n_i = [\sin(\theta)\cdot\cos(\varphi), \sin(\theta)\cdot\sin(\varphi), \cos(\theta)]$

stress vector: $S_i = \sigma_{ij} n_j$

shear stress: $\tau_i = \sigma_{ij} n_j - (n_j\ \sigma_{jk} n_k) n_i$

normal stress: $\sigma_n = n_i \sigma_{ij} n_j$

Shear stress path



The deviatoric stress tensor

- The stress tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

- split in a volumetric and a deviatoric part

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \sigma_{\mathrm{h}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{xx} - \sigma_{\mathrm{h}} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} - \sigma_{\mathrm{h}} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \sigma_{\mathrm{h}} \end{bmatrix}$$

$$= \sigma_{\mathrm{h}} \mathbf{I} + \mathsf{s}^{\mathrm{d}}$$

- the volumetric part contains the hydrostatic stress
- the deviatoric part reflects the shear stress

“Amplitude” of deviatoric stress

$$\sigma^{\mathrm{d}}_{ij,\mathrm{a}}(t) = \sigma^{\mathrm{d}}_{ij}(t) - \sigma^{\mathrm{d}}_{ij,\mathrm{m}}$$

time dependent

with $\sigma^{\mathrm{d}}_{ij,\mathrm{m}}$ from

$$min[max_{(t \in T)}[J_2(\sigma^{\mathrm{d}}_{ij,\mathrm{m}}(t) - \sigma^{\mathrm{d}}_{ij,\mathrm{m}})]]$$

where

$$J_2 \equiv \sigma_{\mathrm{vM}} = \sqrt{\frac{3}{2}\sigma^{\mathrm{d}}_{ij}\sigma^{\mathrm{d}}_{ij}}$$

Shear stress "amplitude"

- We can express the "amplitude" of the Tresca and von Mises stress using this "amplitude" of the deviatoric stress tensor

$$\tau_{\mathrm{Tr,a}}(t) = \frac{\sigma^{\mathrm{d}}_{a,1}(t) - \sigma^{\mathrm{d}}_{a,3}(t)}{2}$$

with

$$\sigma^{\mathrm{d}}_{\mathrm{a},1}(t) = (\sigma^{\mathrm{d}}_{ij}(t) - \sigma^{\mathrm{d}}_{\mathrm{m},ij})_1$$

and

$$\sigma_{\mathrm{vM}}(t) = \sqrt{\frac{3}{2}\sigma^{\mathrm{d}}_{\mathrm{a},ij}(t)\sigma^{\mathrm{d}}_{\mathrm{a},ij}(t)}$$

with

$$\sigma^{\mathrm{d}}_{\mathrm{a},ij}(t) = \sigma^{\mathrm{d}}_{ij}(t) - \sigma^{\mathrm{d}}_{\mathrm{m},ij}$$

Criteria for equivalent stress

- Dang Van

$$\sigma_{\mathrm{EQDV}}(t) = \frac{\sigma^{\mathrm{d}}_{\mathrm{a},1}(t) - \sigma^{\mathrm{d}}_{\mathrm{a},3}(t)}{2} + a_{\mathrm{dv}}\sigma_{\mathrm{h}}(t)$$

with

$$\sigma_{\mathrm{h}}(t) = \frac{\sigma_x(t) + \sigma_y(t) + \sigma_z(t)}{3}$$

- Crossland

$$\sigma_{\mathrm{EQC}}(t) = \sqrt{\frac{3}{2}\sigma^{\mathrm{d}}_{\mathrm{a},ij}(t)\sigma^{\mathrm{d}}_{\mathrm{a},ij}(t)} + c_{\mathrm{C}}\sigma_{\mathrm{h,max}}$$

with

$$\sigma_{\mathrm{h,max}} = max_t(\sigma_{\mathrm{h}}(t))$$

Proportional multiaxial loading

- Proportional loading

  $\sigma_{ij} = a_{ij} + c_{ij} \cdot f(t)$

  a_{ij} and c_{ij} are constants

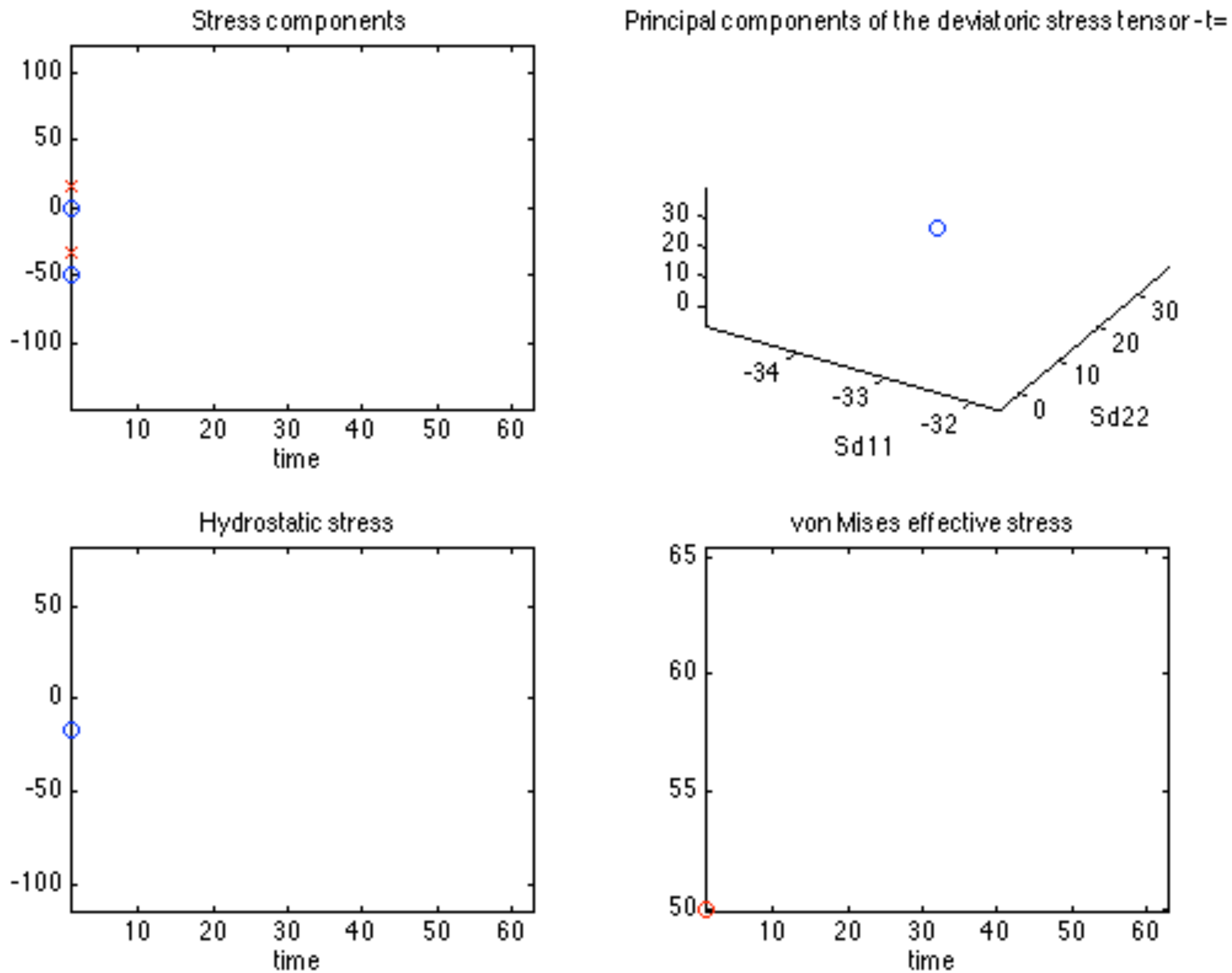
  $f(t)$ is a common time function

- Fixed principal directions:

  the components correspond to a fix direction

$$\sigma_{ij}^{\mathrm{d}}(t) = \begin{bmatrix} \sigma_{xx}^{\mathrm{d}}(t) & \tau_{xy}^{\mathrm{d}}(t) & \tau_{xz}^{\mathrm{d}}(t) \\ \tau_{yx}^{\mathrm{d}}(t) & \sigma_{yy}^{\mathrm{d}}(t) & \tau_{yz}^{\mathrm{d}}(t) \\ \tau_{zx}^{\mathrm{d}}(t) & \tau_{zy}^{\mathrm{d}}(t) & \sigma_{zz}^{\mathrm{d}}(t) \end{bmatrix} = \begin{bmatrix} \sigma_{1}^{\mathrm{d}}(t) & 0 & 0 \\ 0 & \sigma_{2}^{\mathrm{d}}(t) & 0 \\ 0 & 0 & \sigma_{3}^{\mathrm{d}}(t) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^{\mathrm{d}} & 0 & 0 \\ 0 & a_{22}^{\mathrm{d}} & 0 \\ 0 & 0 & a_{33}^{\mathrm{d}} \end{bmatrix} + \begin{bmatrix} c_{11}^{\mathrm{d}} & 0 & 0 \\ 0 & c_{22}^{\mathrm{d}} & 0 \\ 0 & 0 & c_{33}^{\mathrm{d}} \end{bmatrix} \cdot f(t)$$


Stress components
100
50
0
-50
-100
10 20 30 40 50 60
time
Principal components of the deviatoric stress tensor - t= 1
30
20
10
0
-34
-33
-32
Sd11
0
10
20
30
Sd22
Hydrostatic stress
50
0
-50
-100
10 20 30 40 50 60
time
von Mises effective stress
65
60
55
50
10 20 30 40 50 60
time

Shear stress "amplitude"

- For in-phase loading with fixed principal directions (proportional loading), we can express the "*amplitude*" of the Tresca and von Mises stress using the "*amplitude*" of the deviatoric stress tensor

$$\tau_{\text{Tresca,a}}(t) = \frac{\sigma^{\text{d}}_{1,\text{a}}(t) - \sigma^{\text{d}}_{3,\text{a}}(t)}{2} \qquad \sigma^{\text{d}}_{1,\text{a}}(t) = \sigma^{\text{d}}_{1}(t) - \sigma^{\text{d}}_{1,\text{m}}$$

$$\sigma_{\text{vM,a}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma^{\text{d}}_{1,\text{a}} - \sigma^{\text{d}}_{2,\text{a}}\right)^2 + \left(\sigma^{\text{d}}_{2,\text{a}} - \sigma^{\text{d}}_{3,\text{a}}\right)^2 + \left(\sigma^{\text{d}}_{3,\text{a}} - \sigma^{\text{d}}_{1,\text{a}}\right)^2}$$

it can be shown that σ_{a} and $\sigma_{\text{a}}^{\text{d}}$ yields the same results

$$\tau_{\text{Tresca,a}} = \frac{\sigma_{1,\text{a}} - \sigma_{3,\text{a}}}{2} \qquad \sigma_{1,\text{a}} = \frac{\sigma_{1,\max} - \sigma_{1,\min}}{2}$$

$$\sigma_{\text{vM,a}} = \frac{1}{\sqrt{2}} \cdot \sqrt{(\sigma_{1,\text{a}} - \sigma_{2,\text{a}})^2 + (\sigma_{2,\text{a}} - \sigma_{3,\text{a}})^2 + (\sigma_{3,\text{a}} - \sigma_{1,\text{a}})^2}$$

Equivalent stress criteria

- Sines

$$\sigma_{\mathrm{EQS}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{1,\mathrm{a}}^{\mathrm{d}} - \sigma_{2,\mathrm{a}}^{\mathrm{d}}\right)^2 + \left(\sigma_{2,\mathrm{a}}^{\mathrm{d}} - \sigma_{3,\mathrm{a}}^{\mathrm{d}}\right)^2 + \left(\sigma_{3,\mathrm{a}}^{\mathrm{d}} - \sigma_{1,\mathrm{a}}^{\mathrm{d}}\right)^2} + c_{\mathrm{S}}\sigma_{\mathrm{h,mid}} > \sigma_{\mathrm{eS}}$$

- Crossland

$$\sigma_{\mathrm{EQC}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{1,\mathrm{a}} - \sigma_{2,\mathrm{a}}\right)^2 + \left(\sigma_{2,\mathrm{a}} - \sigma_{3,\mathrm{a}}\right)^2 + \left(\sigma_{3,\mathrm{a}} - \sigma_{1,\mathrm{a}}\right)^2} + c_{\mathrm{C}}\sigma_{\mathrm{h,max}} > \sigma_{\mathrm{eC}}$$

- Dang Van

$$\sigma_{\mathrm{EQDV}} = \frac{\sigma_{1,\mathrm{a}} - \sigma_{3,\mathrm{a}}}{2} + c_{\mathrm{DV}}\sigma_{\mathrm{h,max}} > \sigma_{\mathrm{eDV}}$$